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MODULI SPACE FOR INVARIANT SOLUTIONS OF SEIBERG-WITTEN EQUATIONS

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ABSTRACT

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In this work we study the G-invariant solutions of the Seiberg-Witten equations when G is a cyclic group acting on a manifold M, preserving the metric and the orientation. G is assumed to have a lift to principle Spin^c bundle which gives rise to Seiberg-Witten equations in question. It was shown that when the dimension b_+^G of the G-fixed points of harmonic two forms is positive, for a generic choice of an element in this fixed point set, the moduli space of invariant solutions of Seiberg-Witten equations is a compact, smooth and oriented manifold. In case b_+^G is zero, it was shown that there exist a unique singularity which has a compact neighborhood homeomorphic to a cone on a certain projective space. Using the latter case, a version of the theorem of Fintushel and Stern which gives a necessary condition for a Seifert homology 3-sphere occurs as the boundary of a negative definite four manifold whose first cohomology contains no 2-torsion, is proven.

Keywords: Equivariant Seiberg-Witten theory, Equivariant Seiberg-Witten moduli space, Pseudofree orbifold, Seifert homology 3-spheres.

ÖZ

SEIBERG-WITTEN DENKLEMLERİNİN SABİT ÇÖZÜM UZAYI

UĞUZ, MUHİDDİN

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Bu çalışmada Seiberg-Witten denklemlerinin ölçümü koruyan bir devirli G-grubu etkisi altında sabit kalan çözümlerinin oluşturduğu uzay incelenmektedir. G-etkisinin, Seiberg-Witten denklemlerini yazıldığı uzayları veren, temel $\mathrm{Spin}^c(4)$ demetine genişletilebildiği kabul edilmiştir. G-etkisi altında sabit kalan harmonik iki formların oluşturduğu uzayın boyutu, b_+^G, pozitif iken, bu uzaydan seçilebilecek genel bir eleman için Seiberg-Witten denklemlerinin G-etkisi altında sabit kalan çözümlerinin tıkız, yönlendirilebilir, düzgün bir manifold oluşturduğu gösterilmektedir. b_+^G'nin sıfır olduğu durumda ise, bu uzayda tek bir teklil noktasının olduğu ve bu noktanın bir projektif uzay uzerinde koni yapısında olan tıkız bir komşuluğunun bulunduğu ispatlanmıştır. Bu ikinci durumu kullanarak, Fintushel-Stern'ün Seifert homology 3-kürelerinin ne zaman negatif definit ve birinci cohomolojisinde 2-torsion bulunmayan bir 4-boyutlu manifoldun sınırı olabileceği hakkında gerekli şartları veren teoreminin bir versiyonu ispatlanmıştır.

Anahtar Kelimeler: İnvariyant Seiberg-Witten Teorisi, İnvariyant Seiberg-Witten modül uzayı, Yalancı serbest orbifold, Seifert homoloji 3-küreleri.

Dedicate to my father A.Fikret UĞUZ

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CHAPTER I

Introduction

In 1949 Whitehead [3] classified simply connected closed oriented 4-manifolds up to orientation-preserving homotopy equivalence by their intersection form. A proof of this theorem is given in [8], page 103. Later on M. Freedman in 1982 gave homeomorphism classification of closed, simply connected 4-manifolds [7]. His results were expressed in terms of intersection forms. However the classical tools, like intersection forms, were not enough to detect differential structures. During the 1980's, Simon Donaldson used the Yang-Mills equations to study the differential topology of four-manifolds. Using moduli space of connections on an SU(2) bundle, he introduced an invariant which detects differential structures. However, due to the nonlinearity of Yang-Mills equations, to make explicit computations was not easy and substantial analysis was needed. Sometimes, instead of using this invariant, mere use of moduli space of Gauge equivalence classes of connections on an SU(2) or SO(3) bundle itself led to important results. One of these was a well known theorem of Donaldson which states that the only negative definite, unimodular form, represented by a closed, smooth, simply connected four manifold, is the negative of the standard (diagonal) form.(e.g. see S.Donaldson and P.B. Kronheimer, The geometry of four manifolds and Freed-K. Uhlenbeck)

In the fall of 1994, Edward Witten introduced a set of equations which give the main results of Donaldson Theory in a much simpler way. These equations are now known as the Seiberg-Witten equations. These equations were associated to a $\mathrm{Spin}^c(4)$ structure on the manifold in question and they were invariant under the group of bundle automorphisms of the determinant line bundle associated to this $\mathrm{Spin}^c(4)$ structure. This group is called Gauge group. As in Donaldson theory gauge equivalence classes of solutions of Seiberg-Witten equations forms a moduli space and gives important information about the differential topology of the manifold. In fact, a diffeomorphism invariant, called Seiberg-Witten invariant, is introduced using this moduli space(see [9], [13], [4]).

The moduli space of Gauge equivalence classes of the solutions of the perturbed Seiberg Witten equations is compact and in some cases, for a generic perturbation, it is a zero dimensional manifold and hence consists of finitely many points. In this case, Seiberg-Witten invariant is defined to be the algebraic sum of the points in the moduli space counted with the multiplicities according to the orientation.

As in the case of Donaldson Theory, sometimes one can make use of the singularities instead of trying to eliminate them. For instance just by using Seiberg-Witten moduli space (and not Seiberg-Witten invariant) a much simpler proof of the theorem of Donaldson mentioned above can be given(e.g.see

[9] for such a simple proof). In this work, following a similar idea, we shall first construct the moduli space of solutions of Seiberg-Witten equations that are invariant under certain cyclic group action. We shall study the manifold structure and the special structure near singularities, and using this we shall give a version of well known result of Fintushel and Stern which gives a necessary condition for a Seifert homology 3-sphere occurs as the boundary of a negative definite four manifold whose first cohomology contains no 2-torsion, [5]. In their proof, Fintushel and Stern used moduli space of invariant Yang-Mills equation. The use of the moduli space of invariant Seiberg-Witten equations brings some simplification. For instance, instead of dealing with an SO(3)-vector bundle E, we have a line bundle L, here. This makes counting the singularities problem unnecessary. In fact, in Seiberg-Witten case, for each perturbation there is a unique singularity. In [11] Ruan also considered a moduli space of gauge equivalence classes of invariant solutions of Seiberg-Witten equations. His aim was to obtain an invariant.

Let G be a cyclic group of order α. Suppose G acts to preserve orientation on a closed, oriented four dimensional manifold. Choose a G-invariant Riemannian metric and a characteristic G line bundle L. Let us denote the associated principal U(1)-bundle of L by P_L and the associated principal SO(4)-bundle of T^*M by $P_{\mathrm{SO}(4)}$. Let $P_{\mathrm{Spin^c}(4)}$ be the associated principal $\mathrm{Spin^c}(4)$-bundle whose determinant bundle is L. Assume G action on $P_{\mathrm{SO}(4)} \times P_{\mathrm{L}}$ lifts to a G action on $P_{\mathrm{Spin^c}(4)}$. Let $\not{D}_A$ denote the Dirac operator associated to this $\mathrm{Spin^c}(4)$-structure. Since $\not{D}$ is shown to be equivariant, the map $\not{D}^G$ which is the restriction to the G-fixed point set of the domain of $\not{D}$ makes sense. For a Seifert homology 3-sphere $\Sigma(a)$, in Chapter V, an invariant $S(a)$ of $\Sigma(a)$ is defined.

Main theorems of this thesis are the following.

Theorem : If $b_+^G \gneqq 0$, then for a generic perturbation ϕ in Ω_+^G, the moduli space $\mathcal{M}_\phi^G$ of Seiberg-Witten equations perturbed by ϕ is an oriented smooth manifold of dimension $d^G = \mathrm{ind}(\not{D}_\mathrm{A}^\mathrm{G}) - \mathrm{b}_+^\mathrm{G} - 1$.

Theorem : If $b_+^G = 0$, then for a generic $\phi \in {\Omega_+}^G$, $\mathcal{M}_\phi^G$ is a smooth manifold away from a unique singularity defined by Gauge class of a reducible element. This singularity has a compact neighborhood which is a cone on $\mathbb{CP}^{k-1}$ where k is the complex index of $\not{D}_A^G$.

Theorem : Let $\Sigma(a)$ be a Seifert homology 3-sphere oriented as the boundary of $C = \Sigma(a) \times_{S^1} D^2$ where C is positive definite. If $S(a) > 0$ then $\Sigma(a)$ can not bound a negative definite 4-manifold V whose first homology contains no 2-torsion.

This number $S(a)$ is computable by Atiyah-Singer index theorem and Lefschetz fixed point formulas using fixed point data.

The first two theorems are proved in Chapter IV, as Theorems IV.6 and IV.7 The last theorem is proven in Chapter V, as Theorem V.10.

The organization of this thesis is as follows:

In Chapter II, the preliminaries about ordinary Seiberg-Witten theory is presented.

In Chapter III, we prove that, under certain conditions on the given group action on the base manifold, Seiberg-Witten equations are invariant, and moduli space of these invariant solutions is constructed.

In Chapter IV, The compactness of this moduli space, manifold structure and the structure near singularities are studied.

In Chapter V, the results of Chapter IV is applied to Seifert homology spheres to give another version of Fintushel and Stern's result mentioned above.

CHAPTER II

Preliminaries

II.1 Bundle Theory

In this Chapter, we will mainly follow the settings in Chapter 1 of [9].

Definition II.1 Let G be a Lie group. *A principal G-bundle* is a triple $P(M, G, \pi)$ where P is a smooth manifold on which G acts from the right freely, and around each point of the smooth manifold $M = P/G$ there exists a neighborhood U so that, for the projection $\pi : P \to P/G = M$, $P|_U = \pi^{-1}(U) \cong U \times G$ isomorphic as G-spaces. P is called the *total space,* M is called the *base space* and G is called the *structure group.*

Theorem II.2 Isomorphism classes of principal G-bundles over M are in one to one correspondence with the elements of $H^1(M; G)$ and also with the elements of $[M, BG]$, that is, homotopy classes of the maps from M to the classifying space BG.

Definition II.3 Let F be a smooth manifold on which G acts from left. Then given a principal G-bundle $P(M, G, \pi)$ over M, we define $P_F = (P \times F)/\sim$

where $(p, f) \sim (p \cdot g, g^{-1} \cdot f)$. $P_F \to M$ is called a *fiber bundle associated to P with fiber F.*

Definition II.4 As a special case of fiber bundle, defined above, if we take F to be a vector space V and via a representation $\rho : G \to GL(V)$, define a left action of G by $(g, v) \mapsto \rho(g)(v)$. Then the fiber bundle $(P \times V)/\sim$ we get is called a *vector bundle modeled on V* and denoted by $P \times_\rho V$

Theorem II.5 Again as a special case of fiber bundle, take $F = H$ another Lie group with a group homomorphism $\rho : G \to H$. Define a left action of G on H by : $g \cdot h = \rho(g)h$. Then $P_H = P \times_\rho H$ is a principal H-bundle over M

Definition II.6 Given two principal bundles $P_1(M_1, G_1, \pi_1)$, $P_2(M_2, G_2, \pi_2)$ and a Lie group homomorphism $\gamma : G_1 \to G_2$, a map $\varphi : P_1 \to P_2$ is called a *bundle map* if $\varphi(p \cdot g_1) = \varphi(p) \cdot \gamma(g_1)$. Note that φ induces a map on the base spaces $\tilde{\varphi} : M_1 \to M_2$, and we have $\varphi(p_1) \in \pi_2^{-1}(\tilde{\varphi}(\pi_1(p_1)))$ for all $p_1 \in P_1$.

Given $\gamma : G_1 \to G_2$ and a bundle map $\varphi : P_1 \to P_2$, consider the map $P_1 \times_\gamma G_2 \to P_2$ defined by $[p_1, g_2] \mapsto \varphi(p_1) \cdot g_2$. Since $[p_1 \cdot h_1, \gamma(h_1)^{-1}g_2] \to \varphi(p_1 \cdot h_1) \cdot (\gamma(h_1)^{-1}g_2) = \varphi(p_1) \cdot \gamma(h_1) \cdot \gamma(h_1)^{-1} \cdot g_2 = \varphi(p_1) \cdot g_2$, the above bundle map is well defined and hence we have P_2 is isomorphic to $P_1 \times_\gamma G_2$.

Notation II.7 $\Gamma(E)$ denotes the space of *smooth sections* of the bundle: $p : E \to M$. That is, a smooth map $\psi \in \Gamma(E)$ if $\psi : M \to E$ satisfies $p \circ \psi(x) = x$ for all $x \in M$. We usually write $\Gamma(M)$ for $\Gamma(TM)$.

II.2 Connection and Curvature

Definition II.8 *A* connection *on a vector bundle $p : E \to M$ is a map*

$$\begin{aligned} \nabla : \Gamma(M) \times \Gamma(E) &\rightarrow \Gamma(E) \\ (X, \sigma) &\mapsto \nabla_X \sigma = \nabla(X, \sigma) \end{aligned}$$

which satisfies the following properties:

- $\nabla_X(f\sigma + \tau) = (Xf)(\sigma) + f\nabla_X\sigma + \nabla_X\tau$
- $\nabla_{fX+Y}(\sigma) = f\nabla_X\sigma + \nabla_Y\sigma$

where $(Xf)(p) = X(p)f$ is the directional derivative.

An equivalent way of defining a connection on a vector bundle $p : E \rightarrow M$ is using the isomorphism

$$\Gamma(T^*M \otimes E) \cong \Gamma(\mathrm{Hom(TM, E)}) \cong \mathrm{Hom}_{\mathbb{C}^\infty(\mathrm{M})}(\Gamma(\mathrm{TM}), \Gamma(\mathrm{E}));$$

It is a map

$$\begin{aligned} d^E : \Gamma(E) &\rightarrow \Gamma(T^*M \otimes E) \text{ such that;} \\ d^E(f\sigma + \tau) &= (df) \otimes \sigma + f d^E\sigma + d^E\tau. \end{aligned}$$

Note that, after choosing a local trivialization $(U_\alpha, g_{\alpha\beta})$ such that over U_α the bundle is trivial,i.e. $E|_{U_\alpha} = U_\alpha \times \mathbb{R}^m$, any connection restricted to U_α is of the form $d^E|_{U_\alpha}(\sigma_\alpha) = d\sigma_\alpha + w_\alpha\sigma_\alpha$ where σ_α is a section over U_α and w_α is a $m \times m$ matrix of one forms on M. That is

$$d^E \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \end{pmatrix} = \begin{pmatrix} d\sigma_1 \\ d\sigma_2 \\ \vdots \\ d\sigma_m \end{pmatrix} + \begin{pmatrix} w_1^1 & w_2^1 & . & . & w_m^1 \\ w_1^2 & w_2^2 & . & . & w_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1^m & w_2^m & . & . & w_m^m \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \end{pmatrix}.$$

Notation II.9 $\Omega^k(E) = \Gamma(\Lambda^k(T^*M) \otimes E)$.

We may extend the definition of connection d^E to a $\mathbb{R}$-linear map

$$d^E : \Omega^k(E) \to \Omega^{k+1}(E)$$

by tensoring with de Rham complex. For, define

$$d^E(\sigma_1 \wedge \sigma_2) = d\sigma_1 \otimes \sigma_2 + (-1)^k \sigma_1 \wedge d^E \sigma_2$$

where $\sigma_1 \in \Omega^k,\ \sigma_2 \in \Omega^0(E)$.

Definition II.10 Curvature *of a connection* $d^E : \Omega^0(E) \to \Omega^1(E)$ *on* E *is defined to be the* $\mathbb{C}^\infty(M)$*-linear tensor* $d^E \circ d^E : \Omega^0(E) \to \Omega^2(E)$.

Again, over U_α, we have $d^E \circ d^E(\sigma_\alpha) = (dw_\alpha + w_\alpha \wedge w_\alpha)(\sigma_\alpha) = \Omega_\alpha \sigma_\alpha$, where Ω_α is a matrix of two forms.

One final remark about connection and its curvature is about how they transform from U_α to U_β. In order these locally defined connections and their curvature to be well defined globally, on $U_\alpha \cap U_\beta$, we must have:

$$\begin{aligned} w_\alpha &= g_{\alpha\beta} dg_{\alpha\beta}^{-1} + g_{\alpha\beta} w_{\alpha\beta} g_{\alpha\beta}^{-1} \text{ and} \\ \Omega_\alpha &= g_{\alpha\beta} \Omega_\beta g_{\alpha\beta}^{-1}. \end{aligned}$$

Theorem II.11 (Hodge's Theorem): Every de Rham cohomology class on a compact oriented Riemannian manifold M possesses a unique harmonic representative. Thus

$$H^p(M; R) \cong \mathcal{H}^p(M).$$

Moreover, $H^p(M; R)$ is finite dimensional and $\Omega^p(M)$ possesses direct sum decompositions

$$\Omega^p(M) = \mathcal{H}^p(M) \oplus \Delta(\Omega^p(M)) = \mathcal{H}^p(M) \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^p(M)).$$

II.3 The Groups SO(4), Spin(4) and Spin^c(4)

Following [9], we shall consider the quaternions $\mathbb{H}$ as 2×2 complex matrices of the form $Q = \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} = \begin{pmatrix} w & -\bar{v} \\ v & \bar{w} \end{pmatrix}$. With this identification, we have:

$$\tilde{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; \tilde{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; ; \; \tilde{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; ; \; \tilde{k} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \; ,$$

$$Q = \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} = t\tilde{1} + z\tilde{i} + x\tilde{j} - y\tilde{k}$$

and the matrix multiplication agrees with the quaternion multiplication.

Since $det\ Q = t^2+x^2+y^2+z^2 = \langle Q, Q\rangle$-Euclidean dot product, regarding $(t,z,x,y) \in \mathbb{R}^4$ as $t+iz+jx-ky \in \mathbb{H}$, we can identify the unit sphere in $\mathbb{R}^4$ with the special unitary group

$$\mathrm{SU}(2) = \{Q \in \mathbb{H}\ \langle Q, Q\rangle = 1\} = \left\{ \begin{pmatrix} w & -\bar{v} \\ v & \bar{w} \end{pmatrix} \; ; \; det\ Q = 1 \right\}.$$

Definition II.12 Spin(4) $= \mathrm{SU}_+(2) \times \mathrm{SU}_-(2)$, where $\mathrm{SU}_-(2)$ and $\mathrm{SU}_+(2)$ are copies of SU(2).

Definition II.13 SO(4) $= (\mathrm{SU}_+(2) \times \mathrm{SU}_-(2))/\mathbb{Z}_2$.

A typical element of Spin(4) will be represented by (A_+, A_-). We have special orthogonal representation ρ : Spin(4) $\to$ SO(4) $= (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$ $\quad \rho(\mathrm{A}_+, \mathrm{A}_-)(\mathrm{Q}) = [\mathrm{A}_+, \mathrm{A}_-](\mathrm{Q}) = \mathrm{A}_-\mathrm{Q}\mathrm{A}_+^{-1}$. In fact ρ is surjective and since both SO(4) and Spin(4) are compact Lie groups, it induces an isomorphism in the level of Lie algebras and hence $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ is a covering space(double cover).

Elements of Spin(4) can also be represented by the 4×4 matrices $\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$. This representation suggests that we can also consider Spin(4) as a subgroup of $\mathrm{Spin^c}(4)$, where;

Definition II.14 $\mathrm{Spin^c}(4) = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} ; \ \lambda \in \mathrm{U}(1) = \mathrm{S}^1 \right\}$, which also can be defined as $\mathrm{Spin^c}(4) = \dfrac{\mathrm{Spin}(4) \times \mathrm{U}(1)}{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts diagonally.

We have representation

$$\rho^c : \ \mathrm{Spin^c}(4) \to \mathrm{GL}(\mathbb{H})$$

$$\rho^c \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (Q) = (\lambda A_-) Q (\lambda A_+)^{-1}.$$

We also have a group homomorphism:

$$\pi : \mathrm{Spin^c}(4) \to \mathrm{U}(1), \ \text{given by;}$$

$$\pi \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} = \lambda^2.$$

II.4 SO(4), Spin(4) and $\mathrm{Spin^c}(4)$ Structures on a Manifold M

Definition II.15 SO(4) *structure* is a collection $\{(U_\alpha, g_{\alpha\beta}); \ \alpha, \beta \in \Lambda\}$ where U_α is an open cover of orientable 4 manifold M, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(4)$, satisfying the cocycle condition $g_{\alpha\beta} \, g_{\beta\gamma} = g_{\alpha\gamma}$. An alternative way of defining SO(4) structure is specifying a map $f_0 : M \to \mathrm{BSO}(4)$.

Definition II.16 *An associated* Spin(4) *structure to* SO(4) *structure* is a collection $\{(U_\alpha, \bar{g_{\alpha\beta}})\}$, where $\bar{g_{\alpha\beta}} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4)$ satisfying cocycle condition and $\rho \circ \bar{g_{\alpha\beta}} = g_{\alpha\beta}$, where $\rho : \mathrm{Spin}(4) \to \mathrm{SO}(4)$. Alternatively, an associated Spin(4) structure to SO(4) structure is a lifting of $f_0 : M \to \mathrm{BSO}(4)$ to $\tilde{f}_0 : M \to \mathrm{BSpin}(4)$.

From the obstruction theory, we know that the only obstruction for the existence of this lifting, that is, for the existence of Spin(4) structure, i.e. a bundle with structure group Spin(4), associated to the given SO(4) structure on the tangent bundle $T(M)$, is $w_2(TM) \in H^2(M, \mathbb{Z}_2)$.

Let W_+ and W_- be two copies of $\mathbb{C}^2$. Consider the representations ρ_+, ρ_- given by

$$\rho_\pm \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (w_\pm) \;=\; A_\pm w_\pm .$$

Definition II.17 Given a Spin(4) structure, using the above representations ρ_+, ρ_- , we can define new transition functions $\rho_\pm \circ \bar{g_{\alpha\beta}} : U_\alpha \cap U_\beta \to SU_\pm(2)$, to get two new complex bundles also denoted by W_+ and W_-, called *Spinor bundles* and the relation between these bundles and TM is $TM \otimes \mathbb{C} \cong \mathrm{Hom(W_+, W_-)}$.

Therefore a Spin structure determines $TM \otimes \mathbb{C} \cong \mathrm{Hom(W_+, W_-)}$. Moreover if we also have a line bundle L, $TM \otimes \mathbb{C} \cong \mathrm{Hom(W_+ \otimes L, W_- \otimes L)}$, since $L \otimes L^*$ is the trivial bundle.

Given a Spin(4) structure $\{(U_\alpha, \bar{g_{\alpha\beta}})\}$, if we have a line bundle L with transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(1)$ then we can define a $\mathrm{Spin^c}(4)$structure with the transition functions $h_{\alpha\beta} * \bar{g_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow \mathrm{Spin^c}(4)$, where for $x \in U_\alpha \cap U_\beta$ if $\bar{g_{\alpha\beta}}(x) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$ and if $h_{\alpha\beta}(x) = \lambda$ then $h_{\alpha\beta} * \bar{g_{\alpha\beta}}(x) = \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix}$. Note that these maps also satisfy the cocycle condition.

More generally a $\mathrm{Spin^c}(4)$ structure can be defined as

$$\tilde{g_{\alpha\beta}} : U_\alpha \cap U_\beta \rightarrow \mathrm{Spin^c}(4)$$

with cocycle condition. That is we don't need to have a Spin(4) structure or a line bundle in the fist place. Combining this with π we get a complex line bundle, denoted by L^2. Finally, given a $\mathrm{Spin^c}(4)$ structure, associated to it we can define two bundles $W^+ \otimes L$ and $W^- \otimes L$ although L may not exist. $W_\pm \otimes L$ is the bundle whose transition data is $\rho^c_\pm \circ \tilde{g_\pm}$ where $\rho^c_\pm \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_\pm) = \lambda A_\pm w_\pm$. Note that $TM \otimes \mathbb{C} = \mathrm{Hom(W_+ \otimes L, W_- \otimes L)}$.

Definition II.18 We will give the definition of an associated $\mathrm{Spin^c}(4)$ structure to SO(4) and U(1) structure in three equivalent settings;

i- Given an SO(4) structure on $T(M)$ and U(1) structure, i.e, a complex structure, on line bundle L over M, an associated $\mathrm{Spin^c}(4)$ structure is a principal $\mathrm{Spin^c}(4)$ bundle $P \rightarrow M$ such that the associated frame bundles satisfy $P_{SO(4)}(TM) = P \times_{\rho^c} \mathrm{SO}(4)$ and $P_{S^1}(L) = P \times_\pi S^1$, where $\rho^c[A_+, A_-, \lambda](Q) = [A_+, A_-](Q) = A_- Q {A_+}^{-1}$ and $\pi[A_+, A_-, \lambda] = \lambda^2$.

ii- Given two maps $f_0 : M \rightarrow \mathrm{BSO}(4)$ and $f_1 : M \rightarrow \mathrm{BU}(1) = \mathrm{BSO}(2)$ an associated $\mathrm{Spin^c}(4)$ structure is a lift of the map $f = (f_0, f_1) : M \rightarrow \mathrm{BSO}(4) \times \mathrm{BU}(1)$ to $\tilde{f} : M \rightarrow \mathrm{BSpin^c}(4)$.

iii- Given a collection $\{(U_\alpha, g_{\alpha\beta}, h_{\alpha\beta});\ \alpha, \beta \in \Lambda\}$ where U_α is an open cover of orientable 4 manifold M, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(4)$, $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(1)$, both satisfying the cocycle condition, an associated $\mathrm{Spin^c}(4)$ structure is a collection $\{(U_\alpha, s_{\alpha\beta});\ \alpha, \beta \in \Lambda\}$, where $s_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin^c}(4)$, such that $s_{\alpha\beta}$ is mapped to $(g_{\alpha\beta}, h_{\alpha\beta})$ under the map $\mathbb{Z}_2 \to \mathrm{Spin^c}(4) \to \mathrm{SO}(4) \times \mathrm{U}(1)$.

From the obstruction theory, we know that these liftings exist when L is a characteristic line bundle, i.e, $c_1(L) \equiv w_2(TM)$ mod 2, as the only obstruction for these liftings to exist is $w_2(TM \otimes L) \equiv c_1(L) + w_2(TM) \in H^2(M, \mathbb{Z}_2)$.

Note that the assumption M is compact oriented smooth 4 manifold guarantees the existence of $\mathrm{Spin^c}(4)$ structure. Also the assumption that M is simply connected ensures that the liftings considered above are unique.

II.5 Clifford Algebra and the maps ρ and σ

Let $W = W_+ \oplus W_-$. An element of $\mathbb{H}$ can be considered as an element of End(W). Consider the map

Definition II.19

$$\begin{aligned} \rho : \mathbb{H} &\to \text{End(W) defined by,} \\ \rho(Q) &= \begin{pmatrix} 0 & -\bar{Q}^t \\ Q & 0 \end{pmatrix}. \end{aligned}$$

Since

$$\begin{pmatrix} 0 & -\bar{Q}^t \\ Q & 0 \end{pmatrix} \begin{pmatrix} w_+ \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Qw_+ \end{pmatrix} \text{and}$$

$$\begin{pmatrix} 0 & -\bar{Q}^t \\ Q & 0 \end{pmatrix} \begin{pmatrix} 0 \\ w_- \end{pmatrix} = \begin{pmatrix} -\bar{Q}^t w_- \\ 0 \end{pmatrix},$$

we have $\rho(Q) : W_{\pm} \to W_{\mp}$.

The identity:

$$\rho^2(Q) = \begin{pmatrix} 0 & -\bar{Q}^t \\ Q & 0 \end{pmatrix} \begin{pmatrix} 0 & -\bar{Q}^t \\ Q & 0 \end{pmatrix} = \begin{pmatrix} -\bar{Q}^t Q & 0 \\ 0 & -Q\bar{Q}^t \end{pmatrix} = -\langle Q, Q \rangle I$$

gives the Clifford Algebra structure. For this reason $(\mathrm{End(W)}, \rho)$ is called the Clifford Algebra of $(\mathbb{H} \otimes \mathbb{C}, \langle , \rangle)$. Matrix multiplication is the Clifford multiplication, and dim $(\mathrm{End(W)}) = 16$.

Another way of constructing Clifford Algebra is by defining a basis for it. Consider the following matrices:

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tilde{1} \\ \tilde{1} & 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{i} \\ \tilde{i} & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{j} \\ \tilde{j} & 0 \end{pmatrix} \quad , \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{k} \\ \tilde{k} & 0 \end{pmatrix},$$

In fact $e_1 = \rho(\tilde{1}),\ e_2 = \rho(\tilde{i}),\ e_3 = \rho(\tilde{j}),\ e_4 = \rho(\tilde{k})$, where $\rho : \mathbb{H} \to \mathrm{End(W)}$

defined above. These matrices satisfy the following identities:

$$e_i \cdot e_j = -e_j \cdot e_i, \quad {e_i}^2 = -1$$

and the set

$$\{I \ , \ e_i \ , \ \ e_i \cdot e_j; i < j \ , \ e_i \cdot e_j \cdot e_k; i < j < k \ , \ e_1 \cdot e_2 \cdot e_3 \cdot e_4\}$$

form a basis for End(W). Thus we get

$$\mathrm{End(W)} = (\wedge^0(\mathbb{H})\otimes\mathbb{C})\oplus(\wedge^1(\mathbb{H})\otimes\mathbb{C})\oplus(\wedge^2(\mathbb{H})\otimes\mathbb{C})\oplus(\wedge^3(\mathbb{H})\otimes\mathbb{C})\oplus(\wedge^4(\mathbb{H})\otimes\mathbb{C})$$

That is, we have the isomorphism $End(W) \cong \wedge^*(\mathbb{H}) \otimes \mathbb{C}$ of vector space. To have a Clifford Algebra isomorphism, we define a new multiplication on $\wedge^*(\mathbb{H}) \otimes \mathbb{C}$; that is for $w \in \wedge^*(\mathbb{H}) \otimes 1$, define $e_i * w = e_i \wedge w - \imath(e_i)w$ where interior product $\imath(e_i) : \Lambda^k(\mathbb{H}) \to \Lambda^{k-1}(\mathbb{H})$ is defined by $\langle \imath(e_i)w, \theta \rangle \ = \ \langle w, e_i \wedge \theta \rangle$.

In particular we have a vector space isomorphism $\wedge^2(\mathbb{H})\otimes\mathbb{C} \to$The subspace of End(W) generated by $e_i \cdot e_j \ , \ i < j$; defined by $(e_i \wedge e_j) \otimes 1 \mapsto e_1 \cdot e_2$.

Write $\wedge^2(\mathbb{H}) = \wedge^2_+(\mathbb{H}) \oplus \wedge^2_-(\mathbb{H})$ where,

$$\wedge^2_+(\mathbb{H}) = \langle \ e_1 \wedge e_2 + e_3 \wedge e_4, \ e_1 \wedge e_3 + e_4 \wedge e_2, \ e_1 \wedge e_4 + e_2 \wedge e_3 \ \rangle.$$

Since all the basis elements are trace-free and skew hermitian (i.e. $A = -\bar{A}^t$), and since both are 3 dimensional, $\wedge^2_+(\mathbb{H}) \otimes \mathbb{C}$ is just $su(2)$ - traceless skew hermitian endomorphisms of W, that is Lie algebra of $SU(W)$. So we have:

$$\begin{aligned} \wedge^2_+(\mathbb{H}) \otimes \mathbb{C} &\cong su_+(2) \\ \wedge^2_-(\mathbb{H}) \otimes \mathbb{C} &\cong su_-(2) \end{aligned}$$

In terms of Akbulut's [1] setup, that is let P be a principal $\mathrm{Spin}^c(4)$ bundle, so that

$$T^*M = P \times \mathbb{H}/(p,v) \sim \ (\tilde{p}, A_+ v A_-^{-1}) \text{ where } \tilde{p} = p \cdot [A_+, A_-, \lambda] \text{ and}$$

$$W^+(M) = P \times \mathbb{C}^2/(p,v) \sim (\tilde{p}, A_+ v \lambda^{-1}),$$

on the level of bundles, ρ takes the form

$$\begin{array}{rccc} \rho\colon & \Lambda_2^+(M) & \longrightarrow & SU(W^+) \\ & [p, v_1 \wedge v_2] & \longmapsto & \rho([p, v_1 \wedge v_2]) \end{array}$$

where $\rho([p, v_1 \wedge v_2])([p,x]) = [p, Im(v_2\bar{v_1})x]$.

Definition II.20 For $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in W_+$, we define

$$\sigma : W_+ \to \wedge_+^2(\mathbb{H})$$

$$\sigma(\psi) = 2i \text{ Trace free part of } \left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \bar{\psi_1} & \bar{\psi_2} \end{pmatrix} \right)$$

That is:

$$\begin{aligned} \sigma(\psi) &= 2i \begin{pmatrix} |\psi_1|^2 - \frac{|\psi_1|^2 + |\psi_2|^2}{2} & \psi_1 \bar{\psi_2} \\ \psi_2 \bar{\psi_1} & |\psi_2|^2 - \frac{|\psi_1|^2 + |\psi_2|^2}{2} \end{pmatrix} \\ &= i \begin{pmatrix} |\psi_1|^2 - |\psi_2|^2 & 2\psi_1 \bar{\psi_2} \\ 2\psi_2 \bar{\psi_1} & |\psi_2|^2 - |\psi_1|^2 \end{pmatrix}. \end{aligned}$$

In fact

$$\begin{aligned} \sigma(\psi) &= -\frac{i}{2} \sum\nolimits_{i<j} \langle \psi \ , \ e_i e_j \ \psi \rangle e_i \cdot e_j \\ &= -\frac{i}{2} [\langle \psi, e_1 e_2 \psi \rangle (e_1 e_2 + e_3 e_4) + \langle \psi, e_1 e_3 \psi \rangle (e_1 e_3 + e_4 e_2) \ + \\ & \qquad \langle \psi, e_1 e_4 \psi \rangle (e_1 e_4 + e_2 e_3)] \ \in \wedge_+^2, \end{aligned}$$

and satisfies

$$|\sigma(\psi)| \;=\; [\frac{1}{2}\; tr\; (\sigma(\psi)\sigma(\bar{\psi})^t)]^{1/2} \;=\; |\psi|^2.$$

Again, in terms of bundles, σ takes the form:

$$\begin{array}{rccc} \sigma: & W^+ & \longrightarrow & \Lambda_2^+(M) \\ & [p,x] & \longmapsto & [p, \frac{1}{2}(xi\bar{x})]. \end{array}$$

We have adjoint action of Spin(4) and $\mathrm{Spin^c}(4)$ on $End(W)$;

$$Ad \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (T) \;=\; \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} T \begin{pmatrix} (\lambda A_+)^{-1} & 0 \\ 0 & \lambda A_-{}^{-1} \end{pmatrix}.$$

Note that these actions preserve the decomposition of

$\mathrm{End(W)} = \sum_{\mathrm{i}=1}^{4} \Omega^{\mathrm{i}}(\mathrm{M}) \otimes \mathbb{C}$.

Definition II.21 *A Spin connection on W* is a connection which can locally be expressed as $d_A\sigma = d\sigma + \phi_\alpha\sigma$, where ϕ_α are 1-forms taking values in the Lie algebra of Spin(4).

Now since $\mathrm{Spin}(4) = \mathrm{SU}_+(2) \oplus \mathrm{SU}_-(2)$ and Lie algebra $su_\pm(2) = \Omega_\pm^2(M) \otimes \mathbb{C}$, we have Lie Algebra of Spin(4), that is:

$$\mathrm{spin}(4) = \Omega^2(\mathrm{M}) \otimes \mathbb{C} = \langle\; \mathrm{e_i} \cdot \mathrm{e_j};\; \mathrm{i} < \mathrm{j}\; \rangle.$$

Therefore ϕ_α should be of the form $\phi_\alpha = \sum_{i<j} \phi_{\alpha ij} e_i \cdot e_j$ where $\phi_{\alpha ij}$ are skew symmetric ordinary 1-forms, that is $\phi_{\alpha ij} \;=\; -\phi_{\alpha ji}$ since $e_i \cdot e_j \;=\; -e_j \cdot e_i$.

Using this connection, we can define a new connection on End(W) by $(d_A^{\mathrm{Hom}} w)(\sigma) \;=\; d_A(w\sigma) - w d_A\sigma$ for $w \in \Gamma(\mathrm{End(W)})$, $\sigma \in \Gamma(\mathrm{W})$.

Hence, given connection on L, using the Levi-Civita connection on TM, induces a connection on the $\mathrm{Spin^c}(4)$ bundle $P_{\mathrm{Spin^c}(4)}$. Also, any connection d_A

on the bundle W, induces a connection $d_A^{End(W)}$ on the bundle End(W) which in turn restricts to an orthogonal connection $d_A^{End(W)}|_{TM\otimes\mathbb{C}}$ on the tangent bundle $TM \otimes \mathbb{C} \subset \mathrm{End(W)}$.

Finally, before defining the Seiberg-Witten equations and the moduli space, we introduce the Gauge group.

II.6 Gauge Group

Definition II.22 *A gauge transformation* on a line bundle L is a bundle homomorphism $h : L \to L$ commuting with the action of the structure group U(1). That is $h(g \cdot a) = g \cdot h(a) \quad \forall g \in \mathrm{U}(1)$.

The set of all gauge transformations of L form a group, denoted by $\mathcal{G}(L)$, under composition. This group can be considered as maps $f : M \to S^1$, see Section 1.7 of [9] for details. Hence we have $\mathcal{G}(L) \cong \mathrm{Map(M, S^1)}$.

We define an action of the gauge group $\mathcal{G}(L)$ on $\mathcal{A}(L)$ by $g \cdot d_A = d_A + gdg^{-1}$ which can also be expressed as $g \circ d_A \circ g^{-1}$. Action of $\mathcal{G}(L)$ on $\Gamma(W^+)$ is just complex multiplication.

Note that if we regard $\mathcal{G}(L)$ as $\mathrm{Map(M, S^1)}$ then, since M is simply connected, any $g \in \mathcal{G}(L) = \mathrm{Map(M, S^1)}$ can be written as $g = e^{iu}$ for some $u : M \to \mathbb{R}$. According to this representation, $g \cdot (d_{A_0} - ia, \psi) = (d_{A_0} - i(a + du), e^{iu}\psi)$.

Fix a base point $P_0 \in M$ and define $\mathcal{G}_0(L) = \{g \in \mathcal{G}(L);\ g(P_0) = 1\}$.

We have the isomorphism $\mathcal{G}(L) \to \mathcal{G}_0(L) \times S^1$ defined by $h \mapsto (s^{-1}h, s)$ where $s = h(P_0) \in S^1$; $h \in \mathrm{Map(M, S^1)} = \mathcal{G}\mathrm{(L)}$.

Note that $\mathcal{G}_0(L)$ acts freely on $\mathcal{A}(L)$ since $d_A + gdg^{-1} = d_A$ means $gdg^{-1} = 0$ that is $dg^{-1} = 0$ which holds if and only if g =constant. Elements of S^1 are constant functions $M \to S^1$. Hence S^1 acts trivially on $\mathcal{A}(L)$, whereas it acts freely on $(\Gamma(W^+) - 0)$ as complex multiplication.

Definition II.23 *The Dirac operator* is the map

$$\begin{aligned} \not{D}_A : \Gamma(W \otimes L) &\to \Gamma(W \otimes L) \\ \psi &\mapsto \not{D}_A \psi = \sum_{i=1}^{4} e_i \cdot d_A \psi(e_i) = \sum_{i=1}^{4} \rho(e^i)(\nabla_{e_i}\psi), \end{aligned}$$

where $d_A : \Gamma(W \otimes L) \to \Gamma(T^*M \otimes (W \otimes L)) \cong \mathrm{Hom}_{C^\infty(M)}(\mathrm{TM}, \mathrm{W} \otimes \mathrm{L})$, $e_i \in TM \otimes \mathbb{C} \subset \mathrm{End}(\mathrm{W} \otimes \mathrm{L})$ and $\mathrm{e^i} \in \mathrm{T^*M} \otimes \mathbb{C}$ are orthonormal basis, ∇_{e_i} is the covariant derivative along e_i.

II.7 Seiberg-Witten Equations

Let M be oriented, Riemannian 4-manifold with a $\mathrm{Spin^c}(4)$ structure. We consider the pairs (d_A, ψ) where d_A is a connection on L^2 and $\psi \in \Gamma(W_+ \otimes L)$. Let $\mathcal{A} = \{(d_{A_0} - ia, \psi)\}$ denote the configuration space. Recall the maps

$$\begin{aligned} \rho &: \Lambda_2^+(M) \to SU(W^+) \\ \sigma &: \Gamma(W^+) \to \Omega_+^2(M) \\ \not{D}_A^+ &: \Gamma(W^+ \otimes L) \to \Gamma(W^- \otimes L) \end{aligned}$$

We set

$$\begin{aligned} \not{D}_A^+ \psi &= 0 \\ F_A^+ &= i\sigma(\psi) \end{aligned}$$

where $F_A^+ \in \Gamma(\Omega^2(T^*M \otimes i\mathbb{R})) = \Omega^2(M)$.

Notation II.24 $\mathcal{B}(L) = (\mathcal{A}(L) \oplus \Gamma(W^+ \otimes L))/\mathcal{G}(L)$.

Notation II.25 $\mathcal{B}^*(L) = (\mathcal{A}(L) \oplus \Gamma(W^+ \otimes L))/\mathcal{G}(L) - \{[A, 0]\}$.

Notation II.26 $\widetilde{\mathcal{M}}(L)$ denotes the moduli space of $\mathcal{G}_0(L)$ equivalence classes of the solutions of the Seiberg-Witten equations.

Notation II.27 $\mathcal{M}(L)$ denotes the moduli space of gauge equivalence classes of the solutions of the Seiberg-Witten equations. That is

$$\mathcal{M}(L) = \{(A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+ \otimes L);\ \not{D}_A^+ \psi = 0 \text{ and } \mathrm{F}_\mathrm{A}^+ = \mathrm{i}\sigma(\psi)\}/\mathcal{G} = \widetilde{\mathcal{M}}(\mathrm{L})/\mathrm{S}^1.$$

Sometimes we will need to define the perturbed Seiberg-Witten equations and perturbed moduli space. These are the equations

$$\begin{aligned} \not{D}_A^+ \psi &= 0 \\ F_A^+ &= i\sigma(\psi) - \phi \end{aligned}$$

Notation II.28 $\mathcal{M}_\phi(L)$ denotes the moduli space of gauge classes of the solutions of the Seiberg-Witten equations. That is

$$\mathcal{M}_\phi(L) = \{(A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+ \otimes L);\ \not{D}_A^+ \psi = 0 \text{ and } \mathrm{F}_\mathrm{A}^+ = \mathrm{i}\sigma(\psi) - \phi\}/\mathcal{G}.$$

CHAPTER III

Moduli Space of Invariant Solutions of Seiberg-Witten Equations

In this chapter we shall consider a certain cyclic group G acting on M and will show that under certain conditions on M and on this action, Seiberg-Witten equations are invariant.

Given a smooth closed 4-manifold M with a Riemannian metric on it and a characteristic line bundle L over M. Let G be a compact Lie group acting on the base manifold M to preserve the inner product and orientation, also acting on the characteristic line bundle L, commuting with the base projection and mapping fibers directly to fibers as a complex linear map. That is, let L be a G-line bundle. We will also assume that the G-action on L lifts to the associated $\mathrm{Spin}^c(4)$ bundle whose determinant line bundle is L. We will take G a cyclic group of order α. Furthermore, we will also assume that M/G has a positive definite intersection form, and that $H^1(M/G;\mathbb{R}) = 0$. Note that since G is finite and preserves the orientation, M/G is a real homology manifold, that is M/G satisfies Poincare duality with coefficients in $\mathbb{R}$. Hence M/G has a well defined intersection form over $\mathbb{R}$.

We have the following actions:

- G acts on $\mathcal{A}(L)$-space of connections on L:

  For $g \in G$ and $d_A \in \mathcal{A}(L)$ we define $g \cdot d_A = g d_A g^{-1}$,

- G acts on $\Gamma(W \otimes L)$:

  For $g \in G$ and $\psi \in \Gamma(W \otimes L)$ we define $g \cdot \psi = g \psi g^{-1}$,

- G acts on $\mathcal{G}(L)$:

  For $g \in G,\ h \in \mathcal{G}$ we define $g \cdot h = g h g^{-1}$,

- G acts on $T(M)$ and hence on $\Omega^*(\mathrm{M})$:

  For $g \in G$ and $v \in T(M)$ we define $g \cdot v = dg(v)$.

Note that the curvature map $F : \mathcal{A}(L) \to \Omega^2(\mathrm{End(L)})$, defined by $F(d_A) = d_A \circ d_A = F_A$, is equivariant with respect to both G and $\mathcal{G}(L)$ actions. That is $g \cdot F_A = F_{g \cdot A}$ and $h \cdot F_A = F_{h \cdot A}$ for all $g \in G$ and $h \in \mathcal{G}(L)$.

Theorem III.1 Seiberg Witten equations are invariant under G-action.

Proof : First consider

$$\begin{array}{rccc} \rho: & \Lambda_2^+(M) & \longrightarrow & SU(W^+) \\ & [p, v_1 \wedge v_2] & \longmapsto & \rho([p, v_1 \wedge v_2]) \end{array}$$

where $\rho([p, v_1 \wedge v_2])([p, x]) = [p, Im(v_2 \bar{v_1}) x]$.

For any $g \in G$, it satisfies

$$\begin{aligned}
\rho(g\cdot[p, v_1\wedge v_2])([g\cdot p, x]) &= \rho([g\cdot p, v_1\wedge v_2])([g\cdot p, x]) \\
&= [g\cdot p, Im(v_2\bar{v_1})x] \\
&= g\cdot[p, Im(v_2\bar{v_1})x] \\
&= g\cdot\rho([p, v_1\wedge v_2])[p, x] \\
&= g\ \rho([p, v_1\wedge v_2])g^{-1}\ [g\cdot p, x] \\
&= (g\cdot\rho([p, v_1\wedge v_2]))\ [g\cdot p, x].
\end{aligned}$$

hence

$$g\cdot\rho(-) = \rho(g\cdot -);\ \text{that is } \rho \text{ commutes with G-action.}$$

Now consider,

$$\begin{aligned}
\sigma\colon\quad W^+ &\longrightarrow \Lambda_2^+(M) \\
[p, x] &\longmapsto [p, \tfrac{1}{2}(xi\bar{x})]
\end{aligned}$$

For any $g \in G$, we have

$$\sigma(g\cdot[p, x]) = \sigma([g\cdot p, x]) = [g\cdot p, \tfrac{1}{2}(xi\bar{x})] = g\cdot[p, \tfrac{1}{2}(xi\bar{x})] = g\cdot\sigma([p, x]).$$

Hence

$$g\cdot\sigma(-) = \sigma(g\cdot(-));\ \text{that is } \sigma \text{ commutes with G-action.}$$

Thus

$$\begin{aligned}
F_1\colon\quad \mathcal{A}(L)\times\Gamma(W^+) &\longrightarrow \Omega_2^+(M) \\
(A, \psi) &\longmapsto F_A^+ - i\sigma(\psi, \psi)
\end{aligned}$$

is equivariant under the G-action.

Finally the Dirac operator

$$\begin{aligned}
\not{D}\colon\quad \mathcal{A}(L)\times\Gamma(W^+) &\longrightarrow \Gamma(W^-) \\
(A, \psi) &\longmapsto \not{D}_A\psi = \rho(\bigtriangledown_A\psi)
\end{aligned}$$

is also equivariant under the action of the group G. This is because $g \cdot (\not{D}(A, \psi)) = g \cdot \not{D}_A \psi = g(\rho(\bigtriangledown_A \psi))g^{-1} = \rho(g \ \bigtriangledown_A \psi \ g^{-1})$, since ρ is G equivariant. On the other hand, $\not{D}(g \cdot (A, \psi)) = \rho(\bigtriangledown_{gAg^{-1}} g\psi g^{-1})$. Since ρ is an isomorphism, all we need to show is

$$g \ \bigtriangledown_A \psi \ g^{-1} = \bigtriangledown_{(gAg^{-1})} g\psi g^{-1}$$

Here ∇ is the Levi-Civita connection on TM, A is a connection on L and ∇_A is the induced connection on W^+. This statement is not only true for the induced connection but also true for any connection. That is, we claim that: For all $A \in \mathcal{A}(L),\ g \in G,\ \psi \in \Gamma(W^+)$, we have: $(gAg^{-1})(g\psi g^{-1}) = g(A\psi)g^{-1}$. Note that this claim finishes the proof of the theorem. For the proof of the claim, we will show that $(gAg^{-1})(g\psi g^{-1})(x)(v) = g(A\psi)g^{-1}(x)(v)\ \forall x \in M$ and $v \in T_xM$;

Let $\gamma : [0, 1] \to M$ be a curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Now since $W^+ = (P \times \mathbb{C}^2)/\sim$, we can write $g\psi g^{-1} \in \Gamma(W^+)$ as $(g\psi g^{-1})|_{g^{-1}\gamma(t)} = [\widetilde{(g^{-1}\gamma)}(t), \zeta(t)]$ where $(g^{-1}\gamma)(t)$ is a lifting of $(g^{-1}\gamma)(t)$ to P using the given connection, and $\zeta(t)$ is a curve in $\mathbb{C}^2$. Thus

$$(g\psi g^{-1})(g^{-1}\gamma)(0) = [\widetilde{(g^{-1}\gamma)}(0), \zeta(0)]$$

This gives an explicit formula for the left side of the expression in the claim above. That is

$(gAg^{-1})(g\psi g^{-1})(x)(v) = g \cdot [\widetilde{(g^{-1}\gamma)}(0),\ \zeta'(0)] = [g \cdot \widetilde{(g^{-1}\gamma)}(0),\ \zeta'(0)]$ (action is on P).

Now let's write down an explicit formula for the right side of the claim:

$g(A\psi)g^{-1} \in \Gamma(\mathrm{Hom}(\mathrm{TM}, \mathrm{W}^+))$

$(g(A\psi)g^{-1})(x)(v) = g \cdot (A\psi)(g^{-1}x)(v) = g \cdot (A\psi)(g^{-1}x)(g^{-1}v) \in W^+$. The last equation follows from the following fact:

For all $f : T_{g^{-1}}M \to W^+_{(g^{-1}x)}$

$g \cdot f : T_x \to W^+_x$ is defined by $(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$

In our case

$$(A\psi)(g^{-1}x) : T_{g^{-1}}M \to W^+_{(g^{-1}x)}$$

$g \cdot (A\psi)(g^{-1}x) : T_x \to W^+_x$

Hence we have

$(g(A\psi)g^{-1})(x)(v) = g(A\psi)(g^{-1}x)(g^{-1}v)$ Now since γ was chosen to be an integral curve of $v \in T_xM$, $g^{-1}\gamma : [0,1] \to M$ satisfies $g^{-1}\gamma(0) = g^{-1}(x)$ and $\frac{d}{dt}(g^{-1}\gamma)(t) \mid_{t=0} = g^{-1}(v)$. That is $g^{-1}\gamma$ is an integral curve of $g^{-1}(v)$. Therefore $\psi((g^{-1}\gamma)(t)) = [\widetilde{(g^{-1}\gamma)}(t), \zeta(t)]$ giving $(A\psi)(g^{-1}x)(g^{-1}v) = [\widetilde{(g^{-1}\gamma)}(0), \zeta'(0)]$. Thus

$(g(A\psi)g^{-1})(x)(v) = [g \cdot \widetilde{(g^{-1}\gamma)}(0), \zeta'(0)]$.

Hence

$(gAg^{-1})(g\psi g^{-1})(x)(v) = [g \cdot \widetilde{(g^{-1}\gamma)}(0), \zeta'(0)] = (g(A\psi)g^{-1})(x)(v)$

Since all the maps involved are G-equivariant, the Seiberg-Witten equations are invariant under G-action. This completes the proof. $\square$

Now for the map

$$\begin{array}{rlcl} F: & \mathcal{A}(L) \times \Gamma(W^+) & \longrightarrow & \Gamma(W^-) \times \Omega_2^+(M) \\ & (A, \psi) & \longmapsto & (\not{D}_A\psi \ , \ F_A^+ - i\sigma(\psi, \psi)) \end{array}$$

the induced G-actions on the corresponding spaces are equivariant. That is

$$g \cdot F(-) = F(g \cdot -)$$

Finally, before defining the G-invariant moduli space, we recall, Definition 2.22, the gauge group $\mathcal{G}(L) = \mathcal{G}_0(L) \times S^1$ and the action of G on this group. Consider the fixed points of this action. $\mathcal{G}^G(L) = (\mathcal{G}^G{}_0(L) \times S^1)^G = \mathcal{G}_0{}^G(L) \times (S^1)^G = \mathcal{G}^G{}_0(L) \times S^1$.

Now we can define the induced map on the fixed point set

$$\begin{array}{rlll} F^G: & \mathcal{A}(L)^G \times (\Gamma(W^+)^G - 0) & \longrightarrow & \Gamma(W^-)^G \times \Omega_2^+(M)^G \\ & (A, \psi) & \longmapsto & (\not{D}_A\psi, F_A^+ - i\sigma(\psi, \psi)) \end{array}$$

We will consider the G-fixed gauge class of the solution space of F^G. Let $\mathcal{M}(L)^G = \widetilde{\mathcal{M}}(L)^G/S^1$ denote the $\mathcal{G}^G$ classes of G-invariant solution space, where $\widetilde{\mathcal{M}}(L)^G$ is the space of $\mathcal{G}^G{}_0$-invariant solutions.

Now we prove the following fact

Proposition III.2 Fix a connection A_0 on L. Then each element of $\tilde{\mathcal{B}}^G = (\mathcal{A}(L)^G \times (\Gamma(W^+)^G - 0))/\mathcal{G}_0^G$, and hence each element of $\widetilde{\mathcal{M}}(L)^G$ has a unique representation of the form $(d_{A0} - ia\ ,\ \psi)$ where $a \in {\Omega^1}^G$ with the property $\delta a = 0$.

Proof : Recall the Gauge equivalence; $(d_{A_0} - ia, \psi) \sim (d_{A_0} - i(a + du), e^{iu}\psi)$. First we will show the existence

existence : It suffices to show that there exists $u : M \to \mathbb{R}$ such that $\delta(a + du) = 0$ and $e^{iu} \in \mathcal{G}_0^G$. That is $\delta du = -\delta a$, i.e., $\Delta u = -\delta a$. By Stoke's Theorem, we have; $\int_M *\delta a = 0$ (M is closed). So δa is orthogonal to constant functions, i.e. to $\mathcal{H}^0(M)$. Since by Hodge Theorem we have $\Omega^0(M) = \mathcal{H}^0(M) \oplus \mathrm{Im}\ \Delta$, we must have $\delta a \in \mathrm{Im}\Delta$. This proves that there is $u : M \to \mathbb{R}$ such that $\Delta u = -\delta a$. Finally since the action of G commutes with both d and δ, and since $a \in {\Omega^1}^G$ so is e^{iu}. If necessary, after adding a constant to u, we may assume that $u(P_0) = 1$, that is $e^{iu} \in \mathcal{G}_0^G$.

Next we prove the uniqueness

uniqueness : Assume $\langle d_{A_0} - ia_1, \psi_1\rangle \sim \langle d_{A_0} - ia_2, \psi_2\rangle$ and $\delta a_1 = 0 = \delta a_2$. Then $a_1 - a_2 = du$ for some $u : M \to \mathbb{R}$. Now taking the inner product in Ω^1, we get; $\langle a_1 - a_2, a_1 - a_2\rangle = \langle du, a_1 - a_2\rangle = \langle u, \delta(a_1 - a_2)\rangle$. So $a_1 = a_2$. $\square$

As a result of above fact, instead of dealing with ${\mathcal{G}_0}^G$-classes of the solution space of F^G, we can add a new equation to the system; that is consider the solution space of

$$
\begin{aligned}
\not{D}_A^+\psi &= 0 \\
F_A^+ &= i\sigma(\psi) \\
\delta a &= 0
\end{aligned}
$$

where a is the G-invariant one form representing the G-invariant connection A.

Notation III.3 By $\mathcal{M}(L)_\phi^G$ we denote the S^1-classes of the G-invariant solution of the perturbed SW-equations

$$\begin{aligned} \not{D}_A^+\psi &= 0 \\ F_A^+ + \phi &= i\sigma(\psi) \\ \delta a &= 0 \end{aligned}$$

where $\phi \in \Omega_+^{2\ G}$.

That is, we define a new map and still denote it by F^G

$$\begin{aligned} F^G: \ (\mathcal{A}(L)^G \bigcap S) \times (\Gamma(W^+)^G - 0) \times \Omega_+^{2\ G} &\longrightarrow \Gamma(W^-)^G \times \Omega_+^{2\ G} \\ (A, \psi, \phi) &\longmapsto (\not{D}_A\psi, F_A^+ + \phi - i\sigma(\psi, \psi)) \end{aligned}$$

where $S = \{A \in \mathcal{A}(L);\ A = A_0 + ia \text{ and } \delta \mathrm{a} = 0\}$.

With this new notation,

$$\begin{aligned} \mathcal{M}(L)_\phi^G &= (\ \ (F^G)^{-1}(0) \ \bigcap\ (\mathcal{A}(L)^G \bigcap S) \times \Gamma(W^+)^G \times \{\phi\})\ \ )/S^1 \\ &= (F_\phi^G)^{-1}(0)/S^1 \\ &= ((F^G)^{-1}(0) \bigcap (\mathrm{pr}_3)^{-1}(\phi))/\mathrm{S}^1 \end{aligned}$$

Where

$$\begin{aligned} \mathrm{pr}_3 : (\mathcal{A}(\mathrm{L})^{\mathrm{G}} \cap \mathrm{S}) \times (\Gamma(\mathrm{W}^+)^{\mathrm{G}} - 0) \times \Omega_+^{2\ \mathrm{G}} &\rightarrow \Omega_+^{2\ G} \\ (A, \psi, \phi) &\mapsto \phi \end{aligned}$$

is the projection onto the third component. That is, for each perturbation ϕ, the G-invariant Seiberg Witten moduli space is a quotient of a slice of $(F^G)^{-1}(0)$ by S^1.

From now on we will assume that all the spaces we are working on are completed with appropriate Sobolev norms as in Chapter 3 of [9].

CHAPTER IV

Topology of Moduli Space of Invariant Solutions of Invariant Seiberg-Witten Equations.

In this chapter we shall study compactness and the manifold structure on the moduli space, whenever this structure exists, as well as the topology near singularities.

Theorem IV.1 If M is simply connected, then for every choice of G-invariant self dual form ϕ, the moduli space $\widetilde{\mathcal{M}}_{\phi}^{G}(L)$ is compact.

Proof : We know that every sequence of $\mathcal{G}_0$ classes of solutions to the perturbed Seiberg Witten equations has a convergent subsequence. A detailed proof is given in section 3.3 of [9]. Using Proposition III.2, and the continuity of the G-action, we can identify $\widetilde{\mathcal{M}}_{\phi}^{G}(L)$ with a closed subspace of $\widetilde{\mathcal{M}}_{\phi}(L)$. Being a closed subspace of a compact space, $\widetilde{\mathcal{M}}_{\phi}^{G}(L)$ is also compact.

We have the following G-invariant version of the lemma in [9]:

Lemma IV.2 For a generic choice of G-invariant connection A, if the index of the operator ${\not{D}_A^+}^G : (\Gamma(W^+ \otimes L))^G \to (\Gamma(W^- \otimes L))^G$ is nonnegative then ${\not{D}_A^+}^G$ is surjective.

Proof :

Let (A, ψ) be a solution of G-invariant Seiberg-Witten Equations. Then at the points $\psi \neq 0$, the map $\Omega^1 \to \Gamma(W^- \otimes L)$, defined by $a \mapsto a \cdot \psi = \rho(a)(\psi)$, is injective: $a \cdot \psi = 0 \Rightarrow a \cdot a \cdot \psi = 0 \Rightarrow -||a||^2\psi = 0 \Rightarrow ||a|| = 0$, since $\psi \neq 0$. Moreover, since $\dim_{\mathbb{C}}(\Gamma(W^- \otimes L)) = 2 \times 1 = 2$, $\dim_{\mathbb{R}}(\Gamma(W^- \otimes L)) = 4 = \dim_{\mathbb{R}}(T^*M)$, this map $\Omega^1 \xrightarrow{\cong} \Gamma(W^- \otimes L)$ is an isomorphism. In fact, we have this isomorphism on the G-invariant spaces. To see this: take $a \in {\Omega^1}^G$, then $g \cdot (\rho(a)(\psi)) = (\rho(g \cdot a)(\psi)) = (\rho(a)(\psi))$ for all $g \in G$. That is, this map takes G-invariant elements to G-invariant elements. Now take any element of $\Gamma(W^- \otimes L)^G$, say $\rho(a)(\psi)$. We claim that $a \in {\Omega^1}^G$. For, using the isomorphism ρ, all we need to show $\rho(g \cdot a) = \rho(a)$, that is $\rho(g \cdot a)(\psi) = \rho(a)(\psi)$ for all $\psi \in \Gamma(W^- \otimes L)^G$. But $\rho(g \cdot a)(\psi) = g \cdot \rho(a)(\psi) = \rho(a)(\psi)$, since ρ commutes with G action. Hence we have ${\Omega^1}^G \xrightarrow{\cong} \Gamma(W^- \otimes L)^G$.

Recall the map ${\not{D}_+}^G : \mathcal{A}^G(L) \times \Gamma(W^+ \otimes L)^G \to \Gamma(W^- \otimes L)^G$ defined by ${\not{D}_+}^G(A, \psi) = {\not{D}_A^+}^G \psi$. We have $d_{(A,\psi)}{\not{D}_+}^G(a, \psi') = {\not{D}_A^+}^G \psi' - a \cdot \psi$. First we claim that at a irreducible, i.e. G-invariant solution (A, ψ) of the Seiberg-Witten equations with $(\psi \neq 0)$, $d_{(A,\psi)}{\not{D}_+}^G : (\Omega^1)^G \times \Gamma(W^+ \otimes L)^G \to \Gamma(W^- \otimes L)^G$ is surjective. To prove this, let U be a an open set on which ψ is never zero. Since ψ is not identically zero, and continuous, we can find such an open set. On U we have the isomorphism $a \mapsto a \cdot \psi$, as proven above. If

$\sigma \in \Gamma(W^- \otimes L)^G$ is orthogonal to Im $d_{(A,\psi)} \not{D}_+{}^G$, that is if $\langle \not{D}_+{}^G(\psi') - a\cdot\psi, \sigma\rangle = 0$ for all $\psi' \in \Gamma(W^+ \otimes L)^G,\ a \in \Omega^{1G}$, then:

- for $\psi' = 0 \in \Gamma(W^+ \otimes L)^G$, we have $\langle a \cdot \psi, \sigma\rangle = 0$ for all $a \in \Omega^{1G}$ that is $\langle \tau, \sigma\rangle = 0$ for all $\tau \in \Gamma(W^- \otimes L)^G$, by the above isomorphism. Hence $\sigma = 0$ on U.

- for $a = 0 \in \Omega^{1G}$, we have $\langle \not{D}_+{}^G(\psi'), \sigma\rangle = 0 = \langle \psi', \not{D}_-{}^G\sigma\rangle$ for all $\psi' \in \Gamma(W^- \otimes L)^G$. That is $\not{D}_-{}^G\sigma = 0$ on U. But then, by the Unique Continuation Theorem [2], we get $\sigma = 0$ on M.

Hence $d_{(A,\psi)} \not{D}_+{}^G : (\Omega^1)^G \times \Gamma(W^+ \otimes L)^G \to \Gamma(W^+ \otimes L)^G$ is surjective for irreducible G invariant solutions (A, ψ).

Now, by Implicit Function Theorem, we have:

$$N = \{(a, \psi);\ \not{D}_A^{+G}(\psi) = 0,\ \psi \neq 0\}$$

is a submanifold of $\mathcal{A}^G(L) \times \Gamma(W^+ \otimes L)^G$. It's tangent space $T_{(A,\psi)}N =$ Ker $d_{(A,\psi)} \not{D}_+{}^G = \{(a, \psi');\ \not{D}_A^{+G}\psi' - a \cdot \psi = 0\}$. Consider the projection $\mathrm{pr}_1 : N \to \Omega^{1G};\ \ (A, \psi) \mapsto A$. We have $d_{(A,\psi)}\mathrm{pr}_1(a, \psi') = -a$, which is Fredholm and has index greater than or equal to the index of $\not{D}_A^{+G}$. To see this, consider

- $(a, \psi') \in$ Ker $(d_{(A,\psi)}pr_1) \Leftrightarrow (\not{D}_A^{+G}\psi' = 0$ and $a = 0) \ \Leftrightarrow (\psi' \in$ Ker $\not{D}_A^{+G}$ and $a = 0)$. So:

$$\text{Ker } (d_{(A,\psi)}pr_1) = \{(0, \psi');\ \psi' \in \text{Ker } (\not{D}_A^{+G})\} \cong \text{Ker } \not{D}_A^{+G}$$

- $b \in$ Im $(d_{(A,\psi)}pr_1) \Leftrightarrow$ there is (a, ψ') such that $a = b$ and $\not{D}_A{}^{+G}\psi' = -b \cdot \psi \ \Leftrightarrow\ b \in \Omega^{1G},\ b \cdot \psi \in$ Im $\not{D}_A{}^{+G}$. So dim(Im $(d_{(A,\psi)}pr_1)) \geq$

$\dim(\mathrm{Im}\,(\not{D}_A^{+G}))$. Hence:

$$\begin{aligned}
\text{codim }(\mathrm{Im}\,(d_{(A,\psi)}pr_1)) &= \dim\,(\Omega^{1^G}) - \dim\,(\mathrm{Im}\,(d_{(A,\psi)}pr_1)) \\
&\leq \dim\,(\Gamma(W^- \otimes L)) - \dim\,(\mathrm{Im}(\not{D}_A{}^{+G})) \\
&= \dim \text{ coker } (\not{D}_A{}^{+G}).
\end{aligned}$$

Thus, since $\not{D}_A^{+G}$ is Fredholm, so is pr_1. And index of pr_1 is as large as the index of $\not{D}_A^{+G}$.

Now let A be a regular value of pr_1. Then $\mathrm{pr}_1^{-1}(\mathrm{A})$ is a submanifold of dimension larger than or equal to the ind $\not{D}_A^{+G}$. Note that $\mathrm{pr}_1^{-1}(\mathrm{A}) \cup \{0\}$ is the solution space of $\not{D}_A^{+G}\psi = 0$. Finally, to finish the proof, we have two cases to consider:

- The case of Ker $\not{D}_A^{+G} = 0$:

  Then ind $\not{D}_A^{+G} = -\dim$ (coker $\not{D}_{\mathrm{A}}^{+G}$). Since by the assumption ind $\not{D}_A^{+G} \geq 0$, we have coker $\not{D}_A^{+G} = 0$ and hence $\not{D}_A^{+G}$ is surjective.

- The case of Ker $\not{D}_A^{+G} \neq 0$:

  Then there is $\psi \neq 0$, whereas $\not{D}_A^{+G}\psi = 0$. To prove $\not{D}_A^{+G}$ is surjective, we take any $\psi_0 \in \Gamma(W^- \otimes L)^G$ and show it is in the image. Since in this case, we know that $d_{(A,\psi)}\not{D}_+{}^G$ is surjective, there exists (a', ψ') such that $d_{(A,\psi)}\not{D}_+{}^G(a',\psi') = \not{D}_A^{+G}(\psi') - a' \cdot \psi = \psi_0$. Since $\not{D}_A^{+G}$ is linear, all we need to prove is $a' \cdot \psi \in Im\ \not{D}_A^{+G}$. But $a' \in \Omega^{1^G} = \mathrm{Im}\ d_{(A,\psi)}\mathrm{pr}_1 = \{a \in \Omega^{1^G};\ a \cdot \psi \in \mathrm{Im}\ \not{D}_A^{+G}\}$. So there is $\psi'' \in \Gamma(W^+ \otimes L)^G$ with $\not{D}_A^{+G}(\psi'') = a_0 \cdot \psi$ and hence $\psi_0 = \not{D}_A^{+G}(\psi' - \psi'')$.

This completes the proof of $\not{D}_A^{+G}$ is surjective. $\square$

Lemma IV.3 The cohomology groups of the G-equivariant fundamental elliptic complex:

$$0 \longrightarrow \Omega_0^G(M) \xrightarrow{d} \Omega_1^G(M) \xrightarrow{d^+} \Omega_+^G(M) \longrightarrow 0$$

are

$$\mathcal{H}^0(M)^G,\ \ \mathcal{H}^1(M)^G \text{ and } \mathcal{H}_2^+(\mathrm{M})^{\mathrm{G}}$$

of dimensions ${b_0}^G$, ${b_1}^G$ and b_2^{+G}.

Proof :

- To compute the cohomology group at the first stage, take $f \in \Omega_0^G(M) \bigcap \text{Ker } d$. Then $df = 0 = \delta f$. Hence $f \in \mathcal{H}^0(M)^G$

- To compute the cohomology group at the second stage, take $w \in \Omega_1^G(M) \bigcap \text{Ker } d_+^G \bigcap (\mathrm{Im\ d^G})^{\perp}$. Then

  $\forall \eta \quad 0 = \langle w, d\eta \rangle = \langle \delta w, \eta \rangle$, therefore $\delta w = 0$.

  Now by the Stoke's Theorem, we have

  $$\|dw\|^2 = \int_M dw \wedge dw = \int_M d(w \wedge dw) = \int_{\partial M = \emptyset} w \wedge dw = 0$$

  This tells us that $d^+ w = 0 \Leftrightarrow d^- w = 0$ giving $w \in \mathcal{H}^1(M)^G$

- To compute the cohomology group at the third stage, take $w \in \Omega_+^G(M) \bigcap (\text{Im } d^+)^{\perp}$. Then

$$\begin{aligned}
& \langle w, d^+\eta\rangle = 0 && \forall \eta \in \Omega^1 \\
\Rightarrow\ & \langle w, d^+\eta \oplus d^-\eta\rangle = 0 && \text{sincew} \in \Omega_2^+ \\
\Rightarrow\ & \langle w, d\eta\rangle = 0 && \forall \eta \in \Omega^1 \\
\Rightarrow\ & \langle \delta w, \eta\rangle = 0 && \forall \eta \in \Omega^1 \\
\Rightarrow\ & \delta w = 0
\end{aligned}$$

On the other hand

$$\begin{aligned}
dw &= - * \delta * w = - * \delta w \text{ since } w \in \Omega_+^G \\
&= 0
\end{aligned}$$

Therefore $w \in (\text{Im } d^+)^\perp \Rightarrow dw = 0 = \delta w \Leftrightarrow w \in \mathcal{H}_+^2(M)^G$-self dual, G-invariant harmonic 2 forms. $\square$

Lemma IV.4 If L is a complex line bundle over a closed, oriented Riemannian 4 manifold, then any $\phi \in \Omega_+^2(M)^G$ can be realized as a curvature of some unitary G-invariant connection A if and only if $\phi \in \Pi^G = \text{Im } d_+^G$ an affine subspace of $\Omega_+^2(M)^G$ of codimension b_+^G.

Proof : Choose a G-invariant base connection A_0 on L. We have in the general case, hence in the G-invariant case that, for any unitary G- invariant connection A on L, $F_A - F_{A_0} = da$ where a is a real valued G-invariant one form on M. Now we define Π^G as

$$\Pi^G = \text{Im} \left\{ \begin{array}{ccc} \Omega^1 & \xrightarrow{d^+} & \Omega_2^+ \\ & & \\ a & \longrightarrow & F_{A_0}^+ + (da)^+ \end{array} \right\}^G = \text{Im} \left\{ \begin{array}{ccc} \Omega^{1G} & \xrightarrow{d_+^G} & \Omega_2^{+G} \\ & & \\ a & \longrightarrow & F_{A_0}^+ + (da)^+ \end{array} \right\}$$

That is $\Pi^G = \operatorname{Im} d_+^G$, where d_+^G is just d_+ defined on the invariant setting. Hence $\operatorname{codim}(\Pi^G) = \dim(\Omega_2^{+G}/\operatorname{Im} d_+^G) = b_+^G$, as is shown above in Lemma IV.3. $\square$

Theorem IV.5 Let M be a closed, simply connected smooth 4-manifold with a Spin^c-structure. Suppose that if $b_+^G = 0$ then $\operatorname{ind}(\not{D}_A^{+G}) \geq 0$. Then for a generic choice of G-invariant self-dual two form ϕ, $\widetilde{\mathcal{M}}_\phi^G(L)$ is an oriented smooth manifold of dimension $d^G = \operatorname{ind}_{\mathbb{R}}(\not{D}_A^G) - b_+^G$.

Proof : Recall the map

$$\begin{array}{rccc} F^G: & (\mathcal{A}(L)^G \bigcap \operatorname{Ker} d^*) \times \Gamma(W^+ \otimes L)^G \times \Omega_+^{2\,G} & \longrightarrow & \Gamma(W^- \otimes L)^G \times \Omega_+^{2\,G} \\ & (A, \psi, \phi) & \longmapsto & (\not{D}_A\psi, F_A^+ + \phi - i\sigma(\psi,\psi)). \end{array}$$

Differential of F^G at (A, ψ, ϕ) is given by:

$$dF^G{}_{(A,\psi,\phi)}(a, \psi', \phi') = (\not{D}_A^+\psi' - ia\cdot\psi \ , (da)^+ - \sigma(\psi,\psi') - \sigma(\psi',\psi) - \phi').$$

Now, we note that for the irreducible solutions (A, ψ, ϕ) with $\psi \neq 0$, of $F^G = 0$, $dF^G{}_{(A,\psi,\phi)}$ is surjective. Because, we already know that the first component is surjective when $\psi \neq 0$ as shown in the proof of Lemma IV.2. For the second part, take a and ψ' to be zero and let ϕ' vary to cover $\{0\} \times \Omega_+^G$. Thus $dF^G{}_{(A,\psi,\phi)}$ is surjective on each component. To be able to say that it is surjective to the product, since the map under consideration is linear, all we need to show is, after fixing one element from one of the component, say $\phi' = 0$, $dF^G{}_{(A,\psi,\phi)}$ is surjective onto $\Gamma(W^-)^G$. But we have proven this before.

Let $U = \{\phi \in \Omega_+^{2\,G}, \phi = F_A^+ -$curvature of some connection A $\Rightarrow \not{D}_A^{+G}$is surjective$\}$. We have two cases to consider:

In the case $b_+^G > 0$: by Lemma IV.4 we know that U is open and dense.

For the case $b_+^G = 0$: by the assumption that ind ${\not{D}_A^+}^G \geq 0$, again as before, we get U is open and dense.

Hence in each case we have U is open and dense. On the other hand for $\phi \in U$, ${dF^G}_{(A,\psi,\phi)}$ is surjective even if $\psi = 0$. Now, by the Implicit Function Theorem, we have:

$$N = \{(A, \psi, \phi);\ \phi \in U,\ F^G(A, \psi, \phi) = 0\}$$

is a submanifold with

$$\begin{aligned}
T_{(A,\psi,\phi)}N &= \text{Ker } {dF^G}_{(A,\psi,\phi)} \\
&= \{(a, \psi', \phi'); {\not{D}_A^+}^G \psi' = ia \cdot \psi',\ \delta a = 0, \\
&\qquad\qquad d^+ a = \sigma(\psi, \psi') - \sigma(\psi', \psi) - \phi'\} \\
&= \{(a, \psi', \phi'); L(a, \psi') = (0, 0, \phi')\},
\end{aligned}$$

where $L(a, \psi') = ({\not{D}_A^+}^G \psi' - ia \cdot \psi,\ \delta a,\ d^+ a - \sigma(\psi, \psi') - \sigma(\psi', \psi))$. is an elliptic operator from $\Gamma(W^+ \otimes L)^G \oplus {\Omega^1}^G$ to $\Gamma(W^- \otimes L)^G \oplus {\tilde{\Omega^0}}^G \oplus {\Omega_+^2}^G$ where ${\tilde{\Omega^0}}^G$ denotes the space of smooth functions on M which integrate to zero. ($f \in {\tilde{\Omega^0}}^G \Rightarrow \int_M f dVol = 0 \Rightarrow \int_M f(*1) = 0 \Rightarrow \langle f, 1 \rangle = 0 \Rightarrow f \in ({\mathcal{H}^0}^G)^\perp \Rightarrow f \in \text{Im } \delta$, by Hodge Theorem, and $f \in {\tilde{\Omega^0}}^G = \text{Im } \delta \Rightarrow$ there exist $w \in {\Omega^1}^G$ with $f = \delta w \Rightarrow \int_M f dVol = \int_M f(*1) = \langle f, 1 \rangle = \langle \delta w, 1 \rangle = \langle w, d1 \rangle = 0.$)

Finally we claim that

$$\begin{aligned}
\text{pr}_3 : N &\rightarrow {\Omega_+^2}^G \\
\text{pr}_3(\text{A}, \psi, \phi) &= \phi
\end{aligned}$$

is Fredholm. In fact: $(d\mathrm{pr}_3)_{(a,\psi,\phi)}(a,\psi',\phi') = \phi'$. Hence

$$\begin{aligned}
\mathrm{Ker}\ (d\mathrm{pr}_3)_{(a,\psi,\phi)} &= \{(a,\psi',0) \in T_{(A,\psi,\phi)}N\} \\
&= \{(a,\psi',0); L(a,\psi') = (0,0,0)\} \\
&= \mathrm{Ker}\ L.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\ (d\mathrm{pr}_3)_{(a,\psi,\phi)} &= \{\phi' \in {\Omega^2_+}^G; L(a,\psi') = (0,0,\phi') \text{ for some} \\
&\qquad\qquad (a,\psi') \in {\Omega^1}^G \oplus \Gamma(W^+ \otimes L)^G\} \\
&= \mathrm{Im}\ (L) \cap (\{0\} \oplus \{0\} \oplus {\Omega^2_+}^G).
\end{aligned}$$

Also note that, since $(\Gamma(W^- \otimes L) \oplus {\tilde{\Omega}^0}{}^G \oplus 0) \cap (\mathrm{Im}\ L)^\perp = 0$, coker $(d\mathrm{pr}_3)_(a,\psi,\phi)$ has the same dimension as coker L, and since L is Fredholm, so is pr_3 and hence, using the fact that U is open and dense, as proved above, and applying the Sard's Theorem to the map $\mathrm{pr}_3 : N \to {\Omega^2_+}^G$, for ϕ a regular value of pr_3, we get the slice $(\mathrm{pr}_3)^{-1}(\phi)$ is a submanifold of N of dimension equal to the index of pr_3 that is equal to the index of L which in turn ,using the homotopy $L_t(a,\psi') = ({\not{D}^+_A}^G\psi' - ita\cdot\psi,\ \delta a,\ d^+a - it(\sigma(\psi,\psi') + \sigma(\psi',\psi)))$, is equal to the index of L_0, where $L_0 = {\not{D}^+_A}^G \oplus \delta \oplus d^+$.

Finally, since $\mathrm{ind}(d^+) = \dim(\ker d|^+_{\ker d^*}) - \dim(\mathrm{coker}\ d|^+_{\ker d^*}) = b^G_1 - b^G_+ = -b^G_+$, we get $d^G = \mathrm{ind}_{\mathbb{R}}(\not{D}^G_A) - b^G_+$.

As for the orientation, the fact that $\widetilde{\mathcal{M}}^G_\phi(L)$ is orientable can be proven by making obvious modifications in non-equivariant case, see page 79 of [9]. We have Ker $L^G_{(A,\psi)} = T_{(A,\psi)}(\widetilde{\mathcal{M}}^G_\phi(L))$ as we have seen in the previous transversality arguments. Hence, if d is the dimension of the moduli space, we have det $(L^G) = \wedge^d\mathrm{Ker}\ (L^G) = \wedge^d(\widetilde{\mathcal{M}}^G_\phi(L))$ where det (L^G)

denotes the determinant line bundle of the family of Fredholm operators $L^G_{(A,\psi)} : p \mapsto L^G_{(A,\psi)}(p)$. Thus a nowhere zero section of det (L^G) will give an orientation of $\widetilde{\mathcal{M}}^G_\phi(L)$. For the family of operators L_t, for $t \in [0,1]$, defined by $L_t(a,\psi') = ({\not{D}^+_A}^G \psi' - ita \cdot \psi,\ \delta a,\ d^+a - 2t\sigma(\psi,\psi))$, the determinant line bundle det (L^G_t) is defined for every t and depends continuously on t. Thus the bundles det (L^G_t) are all isomorphic and it suffices to construct a nowhere zero section of det (L^G_0), where $L_0 = {\not{D}^+_A}^G \oplus \delta^G \oplus {d^+}^G$. On the other hand we have det $(L^G_0) =$ det $({\not{D}^+_A}^G) \otimes$ det $(\delta^G \oplus {d^+}^G)$. Now, det $({\not{D}^+_A}^G)$ has a nowhere zero section which comes from the orientations of the kernal and cokernal of ${\not{D}^+_A}^G$ defined by complex multiplication, and det$(\delta^G \oplus {d^+}^G)$ inherits a nowhere zero section from an orientation of $\mathcal{H}^G_+(M)$. Thus detL_0 is trivialized, finishing the proof. $\square$

IV.1 Smooth Case

The aim of this section is to prove the following theorem

Theorem IV.6 Let M be a closed, simply connected smooth 4-manifold with a Spin^c-structure. If $b^G_+ > 0$ then for a generic choice of G-invariant self-dual two form ϕ, $\mathcal{M}(L)^G_\phi$ is an oriented smooth manifold of dimension $d^G = \mathrm{ind}_{\mathbb{R}}(\not{D}^G_A) - b^G_+ - 1$.

Proof : The existence of a reducible solution in $\widetilde{\mathcal{M}}^G_\phi(L)$, which causes singularity in $\mathcal{M}(L)^G_\phi$, depends on the condition that $c_1(L)$ contains a connection with $F^+_A = 0$, in turn which occurs only if $\phi \in \Pi^G$-a subspace of Ω^G_+ of codimension b^G_+. Since, by the assumption $b^G_+ > 0$, these singularities are avoidable.

Hence $S^1 \subset \mathcal{G}$ acts freely on $\widetilde{\mathcal{M}}_\phi^G(L)$. Therefore $\mathcal{M}(L)_\phi^G$ is an oriented smooth manifold with dim $(\mathcal{M}(L)_\phi^G) = $ dim $(\widetilde{\mathcal{M}}_\phi^G(L)) - 1 = \mathrm{ind}_{\mathbb{R}}(\not{D}_A^G) - b_+^G - 1$. The orientation of $\mathcal{M}(L)_\phi^G$ is induced from the orientation of $\widetilde{\mathcal{M}}_\phi^G(L)$. □

As a final remark, as in Section 3.6 of [9], the moduli space ${\mathcal{M}(L)_\phi}^G$ depends on the choice of G-invariant Riemannian metric and G-fixed perturbation ϕ. In the case $b_+^G \geq 2$, we can combine the two choices of G-fixed perturbations by a curve in Ω_+^G without passing through the subspace Π^G mentioned in Lemma IV.4. Hence, using Smale's infinite-dimensional generalization of transversality theorem [12], we get $\mathcal{M}_{\phi_1}^G(L)$ and $\mathcal{M}_{\phi_2}^G(L)$, are cobordant manifolds. Similarly, changing the G-invariant Riemannian metric on M yields cobordant moduli spaces.

IV.2 Singular Case

The aim of this section is to investigate the topological structure of the moduli space $\mathcal{M}_\phi^G(L)$ for the case $b_+^G = 0$.

Lemma IV.7 When $b_+^G = 0$ and $H^1(M, \mathbb{R})^G = 0$, for each $w \in \Omega_+^G$, there is unique $A \in {\Omega^1}^G$ with $d^+A = w$ and $d^*A = 0$.

Proof : First we will prove the existence:

Recall the G-equivariant fundamental elliptic complex

$$0 \longrightarrow \Omega_0^G(M) \xrightarrow{d} \Omega_1^G(M) \xrightarrow{d^+} \Omega_+^G(M) \longrightarrow 0$$

Since by the assumption $b_+^G = 0$, we have $d^+(\Omega^1)^G = \Omega_+^G$. That is for any $w \in \Omega_+^G$ there exits $A \in {\Omega^1}^G$ with $d^+A = w$.

Now pick any such A. By the Hodge Theorem we have $A = h + df + (d^+)^*\alpha$ where h is a G invariant harmonic one form, f is a function (uniquely determined up to a constant) and $\alpha \in \Omega_+^G$. Consider $A - df = h + (d^+)^*\alpha \in {\Omega^1}^G$ Note that $d^+(A - df) = d^+A - d^+(df) = w$ (since $-\mathrm{pr}_+ d^2 f = 0$) and $d^*(A - df) = d^*(h + (d^+)^*\alpha) = d^*h + d^*(d^+)^*\alpha = 0$.
Therefore for any $w \in \Omega_+^G$ there exists one form $B = A - df \in {\Omega^1}^G$ with $d^+B = w$ and $d^*B = 0$.

Next, we will prove the uniqueness:
Since ${\mathcal{H}^1}^G = H^1(M, \mathbb{R})^G = 0$ by the assumption, from the above elliptic complex we conclude that Ker $d^+ = 0$ i.e. d^+ is one to one. $\square$

Theorem IV.8 When $b_+^G = 0$, $H^1(M, \mathbb{R})^G = 0$ and ind $({\not{D}_A^+}^G) > 0$, the moduli space ${\mathcal{M}_\phi}^G(L)$ is non-empty smooth manifold away from the unique singularity defined by Gauge class of a reducible element. This singularity has a neighborhood which is a cone on $\mathbb{CP}^{k-1}$ where k is the complex index of ${\not{D}_A^+}^G$.

Proof : Fix a G-invariant base connection A_0. For any other G-invariant connection A, we have $A - A_0 = ia$ for some $a \in {\Omega^1}^G$ and $F_A = F_{A_0} + ida$. So we can identify any G-invariant connection A with $a \in {\Omega^1}^G$ and F_A^+ with d^+a. Now after fixing a G-invariant base connection A_0 that corresponds to a one form a satisfying $d^*a = 0$ and using this identification, we see that

$$\begin{aligned}\mathcal{M}(L)_\phi^G &= \widetilde{\mathcal{M}}_\phi^G(L)/S^1 = \{(A, \psi); d^*A = 0,\ \not{D}_A\psi = 0,\ F_A^+ = \sigma(\psi, \psi) + \phi\}^G/S^1 \\ &= \{(a, \psi);\ d^*a = 0,\ \not{D}_{A_0+ia}\psi = 0,\ d^+a = \sigma(\psi, \psi) + \phi - F_{A_0}^+\ \}^G/S^1.\end{aligned}$$

Now for (a, ψ) with $\psi = 0$, i.e. for any reducible element of $\widetilde{\mathcal{M}}_\phi^G(L)$, we have $d^*a = 0$ and $d^+a = \phi - F_{A_0}^+$ which is invariant. Above, in Lemma IV.7, we showed that there exist a unique such $a \in \Omega^{1^G}$ for any fixed $w \in \Omega^+$. Therefore $\widetilde{\mathcal{M}}(L)_\phi{}^G$ contains a unique reducible solution.

Thus $\mathcal{M}(L)^G_\phi$ is generically a non-empty d^G-dimensional manifold away from the unique singular point.

As for the structure of $\mathcal{M}(L)^G_\phi$ around the unique singular point $[(A,0)]$, recall that $F^G(A,\psi) = (\not{D}^G_A\psi\ ,\ F^+_A - \sigma(\psi,\psi))$ and $\mathcal{M}(L) = (F^G)^{-1}(0)/\mathcal{G}(L)$, where $d^*A = 0$. It's linearization is

$$\begin{aligned} dF^G_{(A,\psi)}(a,\psi') &= (\not{D}^G_A\psi' - a\cdot\psi\ ,\ d^+a - \sigma(\psi,\psi') - \sigma(\psi',\psi)), \text{where } d^*a = 0 \\ &= D^G + Q^G \end{aligned}$$

where

$$\begin{aligned} \mathrm{D}^G_{(A,\psi)}(a,\psi') &= (\not{D}^G_A\psi'\ ,\ d^+a) : \mathrm{Ker}\ d^* \to \Gamma(W_-)^G \times \Omega^G_+ \\ \mathrm{Q}^G_{(A,\psi)}(a,\psi') &= (-a\cdot\psi\ ,\ -\sigma(\psi,\psi') - \sigma(\psi',\psi)). \end{aligned}$$

The formal tangent space at a reducible invariant solution $(A,0)$ is given by

$$\begin{aligned} T = \{(a,\psi');\ \not{D}^G_A\psi' = 0 \ \text{ and } \ d^+a = 0\} &= (\mathrm{D}^G + \mathrm{Q}^G)^{-1}{}_{(A,0)}(0,0) \\ &= (\mathrm{D}^G)^{-1}{}_{(A,0)}(0,0) \end{aligned}$$

But, by Lemma IV.7, there is a unique $a \in \Omega^{1^G}$ with $d^+a = 0 = d^*a$, that is $a = 0$. Hence

$$T = \{(0,\psi');\ \not{D}^G_A\psi' = 0\} = \mathrm{Ker}\ \not{D}^G_A$$

This has even dimension (because $\not{D}^G_A : \Gamma(W^+\otimes L)^G \to \Gamma(W^-\otimes L)^G$ is a map between complex spaces and Dirac operator is elliptic, so Fredholm, and hence has finite dimensional kernel and cokernel), say $2k$.

Note that, using the assumptions: $H^1(X)^G = 0 = H^+(X)^G$, we get ${\Omega^2_+}^G/\text{Im } {d^+}^G \cong {\mathcal{H}_+}^2(M)^G = 0$, giving ${d^+}^G$ is onto, i.e. Coker ${d^+}^G = 0$, and also that $\mathcal{H}^1(X)^G \cong H^1(X)^G = 0 = \text{Ker } {d^+}^G = \{a \in {\Omega^1}^G \, ; \, {d^+}^G a = 0 = d^* a\}$. That is, we get Ker ${d^+}^G = 0 =$ Coker ${d^+}^G$.

Now we have the restriction of the Kuranishi map to the Ker D^G :

$\mathcal{K}^G$: Ker $\mathrm{D^G} \to$ Coker $\mathrm{D^G}$. More explicitly

$$\mathcal{K}^G : \left\{ \begin{pmatrix} a \\ \\ \psi' \end{pmatrix} \text{ such that } {d^+}^G a = 0 = {\not{D}^+_A}^G \psi' \right\}^G \to \text{Coker } {d^+}^G \oplus \text{Coker } {\not{D}^+_A}^G$$

That is

$$\mathcal{K}^G : \left\{ \begin{pmatrix} 0 \\ \\ \psi' \end{pmatrix} \text{ such that } {\not{D}^+_A}^G \psi' = 0 \right\}^G \to \text{Coker } {\not{D}^+_A}^G$$

$\mathcal{K}^G$: Ker ${\not{D}^+_A}^G \to$ Coker ${\not{D}^+_A}^G = 0$ (since ${\not{D}^+_A}^G$ is onto by Lemma IV.2).

Hence $\mathcal{K}^G \equiv 0$ and around the singular point $(A, 0)$, structure of $\widetilde{\mathcal{M}}^G(L)$ is given by $(\mathcal{K}^G)^{-1}(0)$ that is Ker ${\not{D}^+_A}^G$.

Then $\mathcal{M}^G(L) = \widetilde{\mathcal{M}}^G(L) \, / S^1 \cong$ Ker ${\not{D}^+_A}^G / S^1 \cong \mathbb{C}^k / S^1$, which is a cone on $\mathbb{CP}^{k-1}$. $\square$

Remark IV.9

The condition ind $({\not{D}^+_{\mathrm{A}}}^{\mathrm{G}}) > 0$ is equivalent to the condition ind $({\not{D}^+_{\mathrm{A}}}^{\mathrm{G}}) > 1$. Because in case ind $({\not{D}^+_A}^G) = 1$, the component of $\mathcal{M}^G(L)$ containing the unique reducible solution $(A, 0)$ is one dimensional and since $\mathcal{M}^G(L)$ is compact, it must be a closed interval and hence has a boundary, giving a contradiction.

Remark IV.10 In the case ind $({\not{D}^+_{\mathrm{A}}}^{\mathrm{G}}) = 0$ again by the compactness, $\tilde{\mathcal{M}}^G(L)$ consists of finitely many points and since S^1 acts freely on the smooth points, there can not be any smooth point. Hence both $\tilde{\mathcal{M}}^G(L)$ and $\mathcal{M}^G(L)$ consist of single point.

CHAPTER V

An Application to Homology 3-Spheres

In this chapter we will give an alternative theorem and proof, using the topology of the invariant Seiberg-Witten moduli space, to the theorem of Fintushel-Stern [5]. First we introduce the terminology. For more details see Chapter 8 and 9 of [10] and also [5].

Definition V.1 *A Pseudofree S^1-action* is a smooth S^1-action on a smooth $(2n+1)$ manifold such that the action is free except for finitely many exceptional orbits $S^1 z_i$; $i = 1 \cdots n$ with isotropy groups G_i at z_i are $\mathbb{Z}_{a_1}, \cdots, \mathbb{Z}_{a_n}$ where $a = a_1, \cdots, a_n$ are pairwise relatively prime. The total isotopy is the product $\alpha = a_1 \cdots a_n$.

Definition V.2 *A Pseudofree S^1 manifold Q* is an odd dimensional smooth manifold with pseudofree S^1 action.

Definition V.3 *A Pseudofree orbifold $X = Q^5/S^1$* is the quotient of the smooth 5-manifold Q^5 by a pseudofree S^1-action.

Note that the neighborhoods of the isolated singularities of X are cones on lens spaces $L(a_i, b_j)$ where $b_j = \alpha/a_j$.

Notation V.4 $DX = X - \cup_{i=1}^{n} \mathrm{int}(\mathrm{cL}(\mathrm{a_i}, \mathrm{b_j}))$.

Since S^1-action on DX is free, it is classified by an Euler class $e \in H^2(DX, \mathbb{Z})$. Consider

$$0 \to H^2(DX, \partial DX, \mathbb{Z}) \xrightarrow{j^*} H^2(DX, \mathbb{Z}) \xrightarrow{i^*} H^2(\partial DX, \mathbb{Z}) = \mathbb{Z}_\alpha$$

Since tubular neighborhood of an exceptional orbit $S^1 z_i$ with isotropy $\mathbb{Z}_{a_i}$ in Q^5 is $D^4 \times_{\mathbb{Z}_{a_i}} S^1$ which is diffeomorphic to $D^4 \times S^1$, the part of Q^5 over each $L(a_i, b_j)$ is just $S^3 \times S^1$. Hence $i^*(e)$ is unit in $\mathbb{Z}_\alpha$. Here S^1-action on the tube $D^2 \times D^2 \times S^1$ of the exceptional orbit $S^1 z_i$ is $t \cdot (z, w, s) = (zt^{r_i}, wt^{s_i}, st^{a_i})$ where r_i and s_i are relatively prime to a_i. Note that

$$(D^2 \times D^2 \times S^1)/S^1 = (D^2 \times D^2)/\mathbb{Z}_{a_i} = cL(a_i; r_i, s_i) = cL(a_i, b_i)$$

where $r_i {s_i}^{-1} \equiv b_i (\bmod \mathrm{a_i})$.

An example of a pseudofree S^1 manifold is Seifert fibered homology 3-sphere $\sum(a) = \sum(a_1, \cdots, a_n)$; which admits a pseudofree S^1 action with exceptional fibers with isotopy $\mathbb{Z}_{a_1}, \cdots, \mathbb{Z}_{a_n}$; and $\sum(a)/S^1 = S^2$.

Let $H \subset K \subset S^1 = \{z \in \mathbb{Z}; ||z|| = 1\} = \{e^{i\theta}; \theta \in [0, 2\pi]\}$.

Definition V.5

For n-tuple of relatively prime integers $a_1, a_2, \ldots, a_n; (a_i, a_j) = 1$, we define complex n-dimensional representation of H as:

$$\begin{aligned} H &\to \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \\ h &\mapsto h^{a_1} + h^{a_2} + \cdots + h^{a_n}, \text{where;} \\ (h^{a_1} + h^{a_2} + \ldots + h^{a_n})(z_1, z_2, \ldots, z_n) &= (h^{a_1} z_1, h^{a_2} z_2, \ldots, h^{a_n} z_n). \end{aligned}$$

Notation V.6 $K \times_H D^4$ denotes the orbit space of $K \times D^4$ by the action of H defined by $h^{-c} + h^a + h^b$, where c is unit modulo $|H|$. That is:

$$\begin{aligned} K \times_H D^4 &= (K \times D^4)/\sim \\ &= \{[(k,d)]; k \in K, d \in D^4\} \\ &= \{[(k,z_1,z_2)] k \in K,\ z_1,z_2 \in \mathbb{C}\} \\ &= \{[(h^{-c}k, h^a z_1, h^b z_2)] k \in K,\ z_1,z_2 \in \mathbb{C}\}. \end{aligned}$$

Let G be the group of αth root of unity.

Definition V.7 *A restricted G-manifold* is a smooth closed four dimensional manifold M with a smooth G-action such that:

- Either M^G is empty or each component has codimension 2.
- There are n orbits (called singular orbits) $Gx_i \cong G/G_{x_i}$ with $\{|G_{x_i}|\}$ relatively prime and $\prod_{i=1}^n G_{x_i} = |G| = \alpha$. Denote $|G_i| = |G_{x_i}| = a_i$.
- The action is free away from M^G and $\cup_{i=1}^n Gx_i$.

Note that, since M is a 4 dimensional manifold, we have

$$M = N \cup (\cup_{i=1}^n (G \times_{G_i} D^4)),$$

where N is the part of M from which interiors of neighborhoods of the singular orbits removed. G_i acts on $G \times D^4$ via $h^{-c} + h^a + h^b$. Also note that $M^G \subset N$.

Then:

$$M/G = X = DX \cup (\cup_{i=1}^n (G \times_{G_i} D^4)/G) = DX \cup (\cup_{i=1}^n D^4/G_i)$$

where $DX = N/G$.

Finally, since $\partial(DX) = \cup_{i=1}^{n} S^3/G_i$, and since G_i acts freely on S^3, we get

$$H^2(\partial DX) = \oplus_{i=1}^{n} H^2(S^3/G_i) = \oplus_{i=1}^{n} \mathbb{Z}_{a_i} = \mathbb{Z}_{\alpha}.$$

Since DX is a smooth manifold, complex line bundles over it are classified by their Euler classes in $H^2(DX)$.

Definition V.8 *A line bundle over DX is called restricted* if the pull-back of its Euler class under inclusion $i : \partial DX \hookrightarrow DX$, that is $i^*(e(L)) \in H^2(\partial DX) \cong \mathbb{Z}_{\alpha}$, is a generator.

Definition V.9 *A restricted G-line bundle L* over a restricted G-manifold M is a G-line bundle L over M such that for $x \in M$:

- If $G_x = G$, then G_x acts trivially on the fibers L_x.
- If $G_x \neq G$, then G_x acts freely on SL_x-fibers of the unit sphere bundle of L.

Note that line bundles over S^3/H correspond to elements in $H^2(S^3/H) \cong \mathbb{Z}_m$, where $m = |H|$. In addition to this, as the line bundles over $G \times_H S^3$ are of the form $G \times_H (S^3 \times \mathbb{C})$, they extend uniquely to line bundles over $G \times_H D^4$, that is $G \times_H (D^4 \times \mathbb{C})$, (see page 103 of [10]). From this observation we conclude that restricted G-line bundles over M are in one to one correspondence with restricted line bundles over DX.

Let G be a subgroup of S^1 generated by the G_i's. In case of dimQ $= 5$, there is a restricted G-manifold M,section 3 of [5] with n singular orbits such that $M/G = X = Q/S^1$ and

$$Q = O \cup (\cup_{i=1}^{n} S^1 \times_{G_i} D^4)$$

$$M = N \cup (\cup_{i=1}^{n} G \times_{G_i} D^4),$$

where $N/G = DX = Q/S^1$.

One example of pseudofree S^1 manifolds is $Q = SL/G$ where L is a restricted G-line bundle over a restricted G-manifold. In this case $Q/S^1 = M/G = X$.

Note that in addition to the one to one correspondence mentioned above, they in turn in one to one correspondence with pseudofree S^1 manifolds with orbit space $X = M/G$.

Let Σ be a Seifert homology 3-sphere, that is a pseudofree S^1 homology sphere whose orbit is S^2. Then $\Sigma \times D^2$ is a pseudo S^1 manifold of dimension 5, where S^1 acts on D^2 as complex multiplication. The correspondence $S^1 z_i \leftrightarrow S^1(z_i, 0)$ is one to one between the singular fibers of Σ and $\Sigma \times D^2$.

Suppose that $\Sigma(a) = \partial V^4$ for some positive definite, smooth 4-manifold. If necessary, after surgering out the the free part of $H_1(V, \mathbb{Z})$, we may assume $H_1(V, \mathbb{Z}_2) = 0$. Consider the space $X = V \cup (\Sigma \times_{S^1} D^2)$-pseudofree S^1 orbifold. Note that $(\Sigma \times_{S^1} D^2)$ is a mapping cylinder of $\pi : \Sigma \to \Sigma/S^1 = S^2$. Because:

$$\begin{aligned}
\Sigma \times_{S^1} D^2 &= \Sigma \times_{S^1} (D^2 - \{0\}) \cup \Sigma \times_{S^1} \{0\} \\
&= \Sigma \times (0, 1] \cup \Sigma \times_{S^1} \{0\} \\
&= \Sigma \times (0, 1] \amalg_{(x,0)\sim\pi(x)} S^2 \\
&= C(\pi).
\end{aligned}$$

There exists a G-manifold M with $M/G = X$ and a restricted line bundle L over M such that $SL/G = Q$ and $Q/S^1 = X$ where G is the subgroup of S^1 generated by the isotropy groups G_i's, that is $G = \mathbb{Z}_\alpha$-group of α-th roots of unity(see [5]).

Consider the restricted line bundle $L_0 = \mathbb{C} \times_{S^1} Q|_{DX}$ over DX. Recall that

$$DX \hookrightarrow (V \cup C(\pi)) = V \cup (\Sigma \times_{S^1} D^2).$$

Note that S^1 action on V is free, that is, the singularities are in the mapping cylinder part $C(\pi)$, therefore we have $V \hookrightarrow DX$ from which we obtain the restriction $L_0|_V \to V$. But noting the fact that $Q|_{DX} = DX \times S^1$, we see that $L_0|_V = \mathbb{C} \times_{S^1} (V \times S^1) = V \times \mathbb{C}$. Hence, we also have $L_0|_\Sigma \to \Sigma = \partial V$ is the trivial bundle. Now since L_0 is defined over the boundaries of DX, i.e. over S^3/G_i, we have the pullback bundle defined over the cylinder $G \times_{G_i} S^3$ in M. And, since any line bundle over $G \times_{G_i} S^3$ is of the form $G \times_{G_i} (S^3 \times \mathbb{C})$, we can extend $\lambda^* L_0$ to the inside the tube by defining $G \times_{G_i} (D^4 \times \mathbb{C}) \to G \times_{G_i} D^4$. This way we extend $\lambda^* L_0$ to a restricted G-line bundle $\lambda^\sharp L_0$ all of M as a G-bundle where $\lambda : M \to M/G$ is the projection.

Next we claim that the line bundle $\lambda^\sharp L_0$ over M is characteristic. That is there is a $\mathrm{Spin}^c(4)$ structure whose determinant line bundle is $\lambda^\sharp L_0$. In fact, since by the assumption the first homology of V does not contain 2-torsion, V is a Spin(4)-manifold. Thus there is a $\mathrm{Spin}^c(4)$ bundle $P_{\mathrm{Spin}^c(4)}$ over V and hence over Σ. Let W' be the complement of the interiors of the union of the slice neighborhoods of the singular orbits and $W = W'/S^1$. Then $(P_{\mathrm{Spin}^c(4)}|_\Sigma \times_{S^1} D^2)|_{W'}$ is a bundle over $(\Sigma \times_{S^1} D^2)$ away from the interiors of the neighborhoods of singular orbits, that is, over W. Now pull back this bundle by the map λ to get a new G-bundle over $(V \times S^1) \cup W'$ that is over M except the interiors of the tubes. Now using the same technique as above, we extend this G-bundle to all of M as a G-bundle.

Let $\not{D}_A^G$ be the Dirac operator associated to the $\mathrm{Spin}^c(4)$ structure constructed above.

Theorem V.10 Let $\Sigma(a)$ be a Seifert homology 3-sphere oriented as the boundary of $C = \Sigma(a) \times_{S^1} D^2$ where C is oriented as to be positive definite. If $\Sigma(a)$ bounds a negative definite 4-manifold V whose first homology contains no 2-torsion then $\mathrm{ind}_{\mathbb{C}}\ (\not{D}_A^G) \leq 1$.

Proof :

Recall that:

$\mathcal{G} = \mathcal{G}_0 \times S^1$,

$\tilde{\mathcal{B}} = (\Gamma(W^+ \otimes L) \oplus \mathcal{A}(L))/\mathcal{G}_0, \qquad \tilde{\mathcal{B}}^* = \tilde{\mathcal{B}} - [(d_A, 0)],$

$\mathcal{B} = \tilde{\mathcal{B}}/S^1 = (\Gamma(W^+ \otimes L) \oplus \mathcal{A}(L))/\mathcal{G}, \quad \mathcal{B}^* = \mathcal{B} - [(d_A, 0)].$

Let $\partial V = \Sigma$. Then we can form the space $X = V \cup (-C(\pi))$ as mentioned earlier. Let M be the smooth four manifold with $M/G = X$. $\lambda^\sharp L_0$ constructed on M is a characteristic line bundle. Since $b_+^G(M) = \dim \mathrm{H}_+^2(\mathrm{M})^{\mathrm{G}} = \dim \mathrm{H}_+^2(\mathrm{M/G}) = \dim \mathrm{H}_+^2(\mathrm{Q/S^1}) = 0$, by the Theorem IV.8, the G-invariant perturbed moduli space $\mathcal{M}_\phi^G$ contains a unique singular point whose neighborhood is a cone over $\mathbb{CP}^{k-1}$ where k is the complex index of $\not{D}_A^G$. Thus if $\mathrm{ind}_{\mathbb{C}}(\not{D}_A^G) > 1$, since the moduli space is compact with boundary equal to $\mathbb{CP}^{k-1}$, we obtain the fundamental class $[\mathbb{CP}^{k-1}] = 0 \in H_{2k-2}(\mathcal{B}^*, \mathbb{Z})$.

On the other hand, let E denote the associated vector bundle over $\mathcal{B}^*$ corresponding to the S^1 fiber bundle $S^1 \to \tilde{\mathcal{B}}^* \to \mathcal{B}^*$. Denote the pull back of this bundle E over $\mathcal{B}^*$ to the boundary $\mathbb{CP}^{k-1} \subset (\mathcal{M}^*_\phi)^G \subset \mathcal{B}^{*G} \subset \mathcal{B}^*$ of the neighborhood of the unique singular point by $i^*(E)$. Since this is a Hopf fibration

$$S^1 \to S^{2k-1} \to \mathbb{CP}^{k-1},$$

we see that $c_1(i^*E) \neq 0$. Moreover, using the fact that the cohomology of $\mathbb{CP}^{k-1}$ is truncated polynomial algebra, i.e. $H^*(\mathbb{CP}^{k-1}, \mathbb{Z}) \cong \frac{P[c]}{(c^k - 1)}$, we see that $\langle c_1(i^*(E))^{k-1}, [\mathbb{CP}^{k-1}]\rangle = \langle i^*(c_1(E))^{k-1}, [\mathbb{CP}^{k-1}]\rangle = \int_{\mathbb{CP}^{k-1}} c_1(E)^{k-1} \neq 0$, contradicting $[\mathbb{CP}^{k-1}] = 0 \in H_{2k-2}(\mathcal{B}^*, \mathbb{Z})$. $\square$

$\mathrm{ind}_{\mathbb{C}}(\not{D}^G_A)$ can be computed by Atiyah-Singer Index theorem and by Lefschetz type of formula using fixed point data (see Chapter 14 of [6] about how this computations should be carried out). Thus one gets some restrictions on the characteristic classes of V and on the data "a" to have $\partial V = \Sigma(a)$.

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