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# INVESTIGATING ELEMENTARY SCHOOL STUDENTS' REASONING ABOUT DISTRIBUTIONS IN VARIOUS CHANCE EVENTS

by

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ABSTRACT OF THE DISSERTATION

Investigating Elementary School Students' Reasoning about

Distributions in Various Chance Events

by

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Data and chance are two related topics that deal with uncertainty, and statistics and probability are the mathematical ways of dealing with these two ideas, respectively (Moore, 1990). Unfortunately, existing literature reveals an artificial separation between probability and data analysis in both research and instruction, which some researchers (Shaughnessy, 2003; Steinbring, 1991) have already called attention to. In a response to the calls from other researchers (e.g., Shaughnessy, 2003) and recommendations from the National Council of Teachers of Mathematics (NCTM, 2000), this dissertation focused on the notion of distribution as a conceptual link between data and chance.

The goal of this study was to characterize a conceptual corridor that contains possible conceptual trajectories taken by students based on their conceptions of probability and reasoning about distributions. A small-group teaching experiment was conducted with six fourth graders to investigate students' development of probability concepts and reasoning about distributions in various chance events over the course of seven weeks. Each student also participated in pre- and post-interviews to assess their understandings of probability concepts and probabilistic reasoning. The retrospective analysis of eleven teaching episodes focused on children's engagement and spontaneous understandings in the context of the tasks designed to support them.

This study details the landmark conceptions and obstacles students have and the opportunities to support the development of probabilistic reasoning and understanding of probability concepts, such as equiprobability, sample space, combinations and permutations, the law of large numbers, empirical probability, and theoretical probability. Consequently, the results of this study yielded two major findings. First, students' qualitative reasoning about distributions involved the conceptions of groups and chunks, middle clump, spread-out-ness, density, symmetry and skewness in shapes, and "easy to get/hard to get" outcomes. Second, students' quantitative reasoning arose from these qualitative descriptions of distributions when they focused on different group patterns and compared them to each other. In addition, this study showed that students tended to rely on causal reasoning about distributions relevant to real life contexts. They also often provided deterministic and mechanical explanations when investigating random events generated by a physical apparatus.

# CHAPTER 1

# INTRODUCTION

Stochastic ideas and intuitions are widely used in almost every field of our lives, e.g. in sciences, in games, in sports, and in legal cases, when we make decisions under uncertainty. Particularly, probability plays a very important role in dealing with uncertain events in many different ways, from predicting tomorrow's weather to supporting a conclusion by evidence. As people make decisions under the conditions of uncertainty, the knowledge of probability and data analysis becomes of critical importance for ordinary citizens to make judgments in chance situations as well as to make decisions on the basis of numerical information in their lives.

Data and chance[1] are two related topics that deal with uncertainty, and statistics and probability are the mathematical ways of dealing with these two ideas, respectively (Moore, 1990). When the National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) publicly strengthened their emphasis on these topics, probability and data analysis[2] began to be introduced in the school mathematics curriculum at all grade bands. After a decade, to keep up with the rapid changes in the world and to make school mathematics understandable and useable in everyday life situations for students, the NCTM released *Principle and Standards for School Mathematics* (NCTM, 2000) emphasizing that a mathematics curriculum should focus on important mathematics which is useful in a variety of school, home, and work settings. Thus, data analysis and probability strand

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[1] Throughout the dissertation, the term "chance" is used to refer to a broader range of ideas and applications of probability while the term "probability" refers particularly to the formalizations involved in assigning probability values to the events (Konold, 2006, personal communication).

[2] The term "data analysis" is used to refer to the mathematical content strand in the school curriculum that deals with the ideas of statistics.

have become one of the content standards students should learn from Pre-kindergarten through Grade 12.

Unfortunately, existing literature reveals an artificial separation between probability and data analysis in both research and instruction, which some researchers (Shaughnessy, 2003; Steinbring, 1991) have already called attention to. It is true that there is a body of research in statistics education that links data and chance in the context of sampling, sampling distribution, and variation in various sampling tasks (e.g., Saldanha & Thompson, 2002; Shaughnessy, Watson, Moritz, & Reading, 1999; Watson & Moritz, 2000). However, this is because probability was foundational in developing the ideas of sampling and sampling distribution and could not be eliminated. Because these are treated as more advanced topics, students in K-12 are introduced to the statistics and probability as isolated and relatively independent topics in mathematics instruction and curriculum.

In the traditional approach, probability and data analysis are thought as very separate topics. One teaches probability starting with the counting principles and Kolmogorov's axioms[3] and maybe teaches with the frequentist approach[4] in which central tendency is discussed as a way of looking at data distribution that is simply a display of the actual outcomes, rather than as probabilistic entities. In data analysis instruction, one teaches about the ideas of center, spread, and shape of data which are not necessarily probabilistic. This separation is also found in the elementary school mathematics curriculum, where the tendency is to have emphasis on number concepts,

[3] Axioms of probability: (1) Probability of an event is a number between 0 and 1. (2) The probability of an event that is the whole sample space is 1. (3) For any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities.
[4] An experimental approach to probability based in the limiting relative frequency of occurrences of an event in an infinite number of trials (Konold, 1991).

such as whole numbers, decimals, percents, fractions, and rational numbers, followed by data collection and analysis, particularly the descriptive statistics concepts (i.e., minimum, maximum, range, mean, mode, and median) and the graphical representations (i.e., line plots, tally charts, bar graphs, and line graphs). In this traditional approach, the conceptual link between probability and data analysis is not treated until the discussion of statistical inference (in advanced levels) in which the idea of probability is imposed.

My study challenges this traditional treatment of probability and data analysis topics and examines the role of the notion of probability distribution as a way to integrate these topics together in early grades. This idea builds on previously conducted studies which reveal two treatments of distribution: (1) A view of data as aggregate within statistical reasoning about distributions and comparing distributions in data analysis (e.g., Cobb, 1999; Lehrer & Schauble, 2000; Hancock, Kaput & Goldsmith, 1992); and (2) A view of distribution across different outcomes as a sample space associated with probabilities within probability distribution and probabilistic reasoning (e.g., Horvath & Lehrer, 1998; Vahey, 1997; Vahey, Enyedy, & Gifford, 2000; Wilensky, 1997). The important distinction between these two kinds of the use of distribution is that the former deals with the position of the most common outcome and the shape, and not probability, whereas the latter refers to the notion of probabilities assigned to the variety of outcomes of an event.

What is discussed above led me to raise the following questions: At how young of an age can children actually begin to make a shift in their reasoning about data with their ideas about probability? Can they be introduced to the notion of distribution in such a way that they engage with ideas of probability as they engage with data as aggregates, rather than teaching them about distributions and then later on trying to inject probability back into it?

## Research Focus

There are extensive research studies in probability and a growing body of research in statistics that document difficulties and misconceptions of students as well as students' conceptions of probability and data analysis topics (see the comprehensive review of these studies: Borovnick & Peard, 1996; Garfield & Ahlgren, 1988; Kapadia, & Borovcnik, 1991; Konold & Higgins, 2003; Shaughnessy, 1992; Shaughnessy, 2003). Given the superficial division between the discussions of probability and data analysis in the research and instruction discussed above, this study was conducted as an attempt to understand the role of the notion of distribution as a link between the data and chance and contribute to the existing literature by addressing the gap.

Through a small-group teaching experiment with fourth-grade students in which I was the teacher-researcher, I investigated students' reasoning about distributions in various chance events supported by the design of a sequence of tasks. The focus of the study is to examine individual students' development through the teaching experiment as they interacted with the teacher-researcher and the other students in their groups. The open-ended tasks which were piloted prior to the study particularly focused on the centered distributions. Students explored these distributions in a variety of ways, such as using chance devices and conducting simulations of uncertain events that can be modeled by a binomial probability distribution in a computer environment.

Given the focus of the study, the main research question that guides the investigation of the fourth-grade students' reasoning about distributions in chance events is: How do students develop reasoning about distributions when engaging in explorations of chance situations through a sequence of tasks, in which students were asked to provide predictions and explanations during the experiments and simulations with objects, physical apparatus, and computer environment?

## Outline of Dissertation

The dissertation is organized into eight chapters. Chapter 1 introduces the study through a brief overview. The problematic that set up the need for this study, the focus of this study, and the research questions are presented in this chapter. Chapter 2 reviews the literature that provided the framework for the study. The selected literature in the second chapter focuses on the emergence of the probability concept; students' misconceptions or heuristics under uncertainty; students' development of probability concepts; the models of students' probabilistic reasoning; and students' reasoning about distributions in the context of data analysis. Chapter 3 presents the theoretical framework and methodology of the study with the description of the research design, the constructivist and socio-cultural philosophies, the participants, the study instruments and tasks, the data gathering process, and the method of analysis. Chapter 4 presents the pilot study results and the revision of the tasks used in the teaching experiment study based on these results. Also, it discusses the development of conjecture that guided the design study. Chapter 5 includes the analysis of the pre-interviews with each participant. Chapter 6 presents the detailed description of students' reasoning about distributions in chance events during the teaching experiment sessions. Chapter 7 includes the analysis of the post-interviews with each student. Finally, Chapter 8 discusses the findings of the design study based on the research question. Also, it presents the limitations of the study, the implications for research and practice, and for the future research.

# CHAPTER 2

# REVIEW OF LITERATURE

In this chapter, I begin to review relevant literature on probability that provided the framework for my study. In reviewing the literature regarding the reasoning about chance, it is necessary and helpful to consider the historical roots of the probability concept before addressing the contributions of the empirical work in probability. Afterward, I provide an overview of the selected research on different kinds of thinking in the acquisition of probabilistic knowledge, students' conceptions of probability, and models of students' probabilistic reasoning. Then, I present the body of research in data analysis that discusses the notion of distribution in those studies. The chapter concludes with the key aspects of the literature and with the statement of the purpose of this study.

## Historical Roots of the Probability Concept

The early ideas of probability, chance, and randomness have existed since ancient times. The use of astragali (a heel bone in animals) and then the early form of die and drawing lots (e.g., straws of unequal length) in gaming, divination, and fortune telling are indications of those ideas (David, 1962). However, the probability concept (as a quantifiable means to describe likelihood) did not emerge until the mid 1600s (Hacking, 1975). The concept of probability itself has developed rather recently when the probabilistic ideas were applied to make decisions in a variety of contexts including the games of chance, law, and life insurance. Prior to this, the word probability was associated with two important ideas, namely *scientia* (knowledge) and *opinio* (opinion) (Hacking, 1975). While *scientia* refers to knowledge of universal truths as well as demonstrable knowledge, in *opinio* probability means approvability of an opinion by some authority or through God-given signs in nature.

A concept of evidence was key to the development of probability concept from the medieval periods to the modern era. In his discussion of evidence, Hacking (1975) distinguishes between two types of evidence: (1) external evidence, having to do with the evidence of testimony, and (2) internal evidence, having to do with the evidence of things. He notes some concepts of evidence that have been around during the Renaissance. Those included the concept of testimony and authority “as the basis for the old medieval kind of probability that was an attribute of opinion” (ibid, p. 32). However, inductive (internal) evidence did not exist until the seventeenth century. The formation of the inductive nature of the evidence evolved from the concept of sign as evidence in *opinio* (Hacking, 1975). Prior to this transformation, the sign-as-evidence indicated with the concept of probability regarded as a matter of testimony by some authority, such as the church. Hacking noted the connection between sign and probability in a sense that signs had probabilities considered more probable than another as they came from the ultimate authority of nature. Moreover, probable signs through which nature gives testimony encompass a spectrum of degrees of evidence from “imperfect” to “very often right” and hence could be both suggestive (i.e., smoke and fire) and predictive (i.e., drawing lots and reading the future). This is to say, “on the one hand, a sign considered as testimony made an opinion probable; on the other, the predictive value of a sign could be measured by the frequency with which the prediction holds” (Hald, 2003, p. 31). According to Hacking (1975), the emergence of the new concept of inductive evidence (empirical evidence) thus resulted in the recognition of the connection between natural signs (i.e., probability as testimony with the approval of data) and the frequency of their correctness (i.e., probability as stable frequencies in repeated trials).

In relation to this transformation from the old concept of sign to the inductive evidence, Hacking notes the duality of the concept of probability that emerged around 1660. More specifically, this dual property of probability, which still exists, is known as (1) epistemic notion of probability, referring to support by evidence (i.e., since his breathing had become shallow and some of his organs had begun to fail, the doctors say he is close to death.) and (2) statistical notion of probability, concerning with stable frequencies of occurrences or certain outcomes (i.e., the probability of getting heads when you toss a coin repeatedly many times gets closer to 0.5). According to Hald (2003), epistemic probabilities apply to "measuring the degree of belief in a proposition warranted by evidence which need not be of a statistical nature" (p.28) while statistical probabilities refer to "describing properties of random mechanisms or experiments, such as games of chance, and for describing chance events in populations, such as the chance of a male birth or the chance of dying at a certain age" (p. 28). Moreover, in Hald's characterization of these two notions of probability, *subjective probabilities* are related to our imperfect knowledge or judgment whereas *objective probabilities* are based on symmetry of games of chance, such as equally possible outcomes, or the stability of relative frequencies in the long run.

An implication of the dual nature of probability mentioned above is twofold. On the one hand, the epistemic or subjective notion of probability emphasizes personal probability relative to our background knowledge and beliefs and, thus, enables us to represent *learning from experience* (i.e., new evidence affect what we know and believe, so we can represent our degrees of belief by coherent personal probabilities that satisfy the basic rules of probability) (Hacking, 2001). On the other hand, the statistical or objective notion of probability underlines stable and law-like regularities in relation to physical and geometrical properties of chance setups and events by calculating relative frequencies from experiments (Hacking, 2001).

According to Steinbring (1991), "beginning with very personal judgments about the given random situation, comparing the empirical situation with their intended models will hopefully lead to generalizations, more precise characterizations, and deeper insights" (p.146). In other words, Steinbring suggested that subjective probabilities based on our knowledge, but not simply a matter of opinion, could be checked by experiment. This interpretation of subjective probabilities was similar to the use of the term 'subjective' among the physicists as well as in Jacques Bernoulli's important distinction between the subjective and objective probabilities which revolutionized the probability theory (Hacking, 1975). In relation to this revolution, Hacking pointed out, the main contribution of Bernoulli was his discussion and extension of the probability concept and its applications in his book, *Ars Conjectandi*, published in 1713 (Hald, 2003). In particular, Bernoulli distinguished between "the probabilities which can be calculated a priori (deductively, from considerations of symmetry) and those which can be calculated only a posteriori (inductively, from relative frequencies)" (Hald, 2003, p. 247). Furthermore, in his book Bernoulli proved the first limit theorem of probability as an attempt to show the applicability of the calculus of probability to other fields where relative frequencies are used as estimates of probabilities. In doing so, he approached the question of whether there was a theoretical counterpart (statistical model) to the empirical outcomes (observations) (Hald, 2003). A formulation of Bernoulli's theorem is the following:

> Consider $n$ independent trials, each with probability $p$ for the occurrence of a certain event, and let $s_n$ denote the number of successes. According to probability theory, $s_n$ is binomially distributed… [*And*] $h_n$ [$h_n = s_n/n$, *the relative frequency*] converges in probability to $p$ for $n \to \infty$. (Hald, 2003, p. 258)

The theorem basically says that as the number of trials gets larger, the difference between the theoretical probability and the empirical result becomes smaller.

An educational analysis of the distinction between the subjective and objective probabilities and its formulation in the Bernoulli's theorem can provide a rich context to discuss the probability and statistics with an emphasis on the conceptual link between these two topics. For instance, the following model (Figure 1) depicts both the ideas discussed in Hacking's historical accounts for the emergence of the probability concept and the implications of the Bernoulli's Theorem (Kazak & Confrey, 2005).

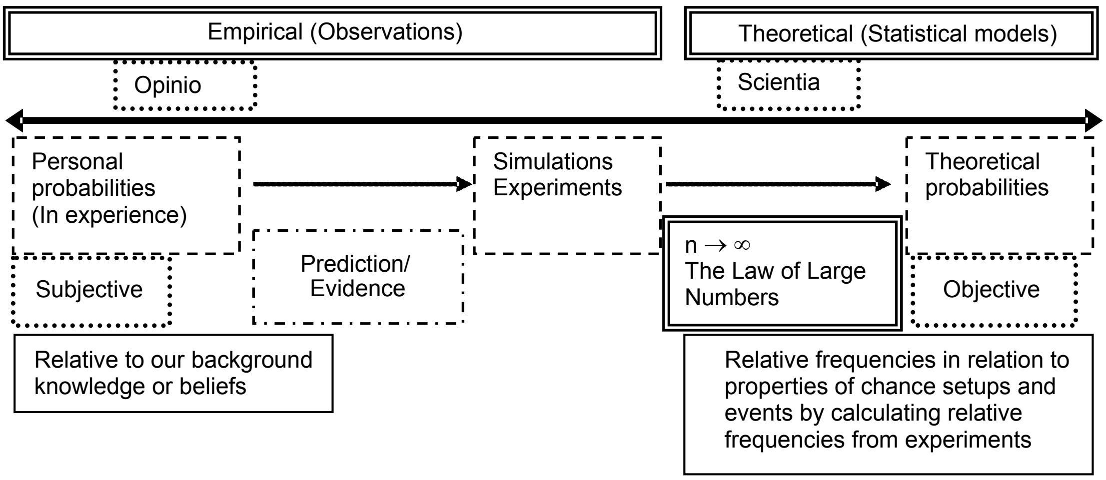


Figure 1. A model to link the discussions of probability and statistics.

This model suggests that when we talk about probabilities, we draw upon a variety of evidence, such as personal knowledge or belief, empirical results, and theoretical knowledge. It further suggests that as one learns to appeal to evidence and create and run simulations, one begins to link *opinio* to *scientia*. Especially, young students' understandings of probability are based on their personal and experiential knowledge about the world. Therefore, the idea of simulation of probability experiments is key to this study as a way to link empirical results to theoretical outcomes. Also, Bernoulli's Theorem is a mediator between the empirical data and the theoretical probabilities through the law of large numbers.

## Different Kinds of Reasoning under Uncertainty

An extensive body of research has primarily identified and documented different types of thinking when making inference or judgment about an uncertain event, which are often called as misconceptions, heuristics, intuitions, or beliefs, across a wide range of age groups from young children to college students. Those include the representativeness heuristic, the availability heuristic, the outcome approach, the law of small numbers, the illicit use of the proportional model, and the equiprobability bias (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; Fischbein & Schnarch, 1997; Kahneman & Tversky, 1972; Tversky & Kahneman, 1973; Tversky & Kahneman, 1982; Konold, 1991; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Lecoutre, 1992). Next, I describe each of these different ways of reasoning under uncertainty as they relate to my study.

**Representativeness Heuristic**

The research of Kahneman and Tversky and their colleagues has shown that people often use certain heuristics-"rapid and more or less automatic judgmental rules of thumb" (Nisbett, Krantz, Jepson, & Kunda, 1983, p. 340)-in judging the likelihood of uncertain events. It appears that people replace the principles of probability theory by heuristics for reasoning under uncertainty. The representativeness heuristic (Kahneman & Tversky, 1972) is one of them.

According to Kahneman and Tversky (1972), the heuristic of representativeness implies that people often evaluate the probability of an uncertain event based on the degree to which it represents some essential features of its parent population. In other words, the probability of an event A is seen higher than that of an event B whenever A is assumed more representative than B. In order to investigate this particular heuristic in detail, Kahneman and Tversky conducted a questionnaire type study with approximately 1500 students in grades 10, 11, and 12 (ages fifteen to eighteen). Researchers then discussed the notion of representativeness in various contexts.

For example, the effect of similarity of sample to population was studied when students were asked the following question:

> All families of six children in a city were surveyed. In 72 families the exact order of births of boys and girls was GBGBBG. What is your estimate of the number of families surveyed in which the exact order of births was BGBBBB? (Kahneman & Tversky, 1972, p.432)

Although the two birth sequences are equally likely, the majority of the students evaluated the BGBBBB sequence to be less likely than GBGBBG. One possible explanation for this response was that the sequence BGBBBB might appear less representative of the proportions of boys and girls (50-50 distribution) in the population

(Kahneman & Tversky, 1972). As a follow-up question, students were asked to estimate the frequency of the sequence BBBGGG in the same context. Then, the student responses showed that it was seen significantly less likely than GBBGBG. However, all three sequences, certainly, are equally likely to occur based on the theoretical model of assigning probabilities. Researchers then claimed that the sequence BBBGGG seemed to the students less random in terms of irregularity in the sequence.

Kahneman and Tversky (1972) also examined the representativeness prediction concerning the sample size conducting an additional experiment with 97 undergraduate students with no background in statistics or probability. For instance, one of the problems given to the students is the following:

> A certain town is served by two hospitals. In the large hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50% of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than 50%, sometimes lower.
>
> For a period of one year, each hospital recorded the days on which (more/less) than 60% of the babies born were boys. Which hospital do you think recorded more such days? (Kahneman & Tversky, 1972, p.443)

To control the response from being biased, the problem was given in two forms. While half of the students were asked to evaluate whether an outcome that is *more* extreme than the specified critical value is more likely to occur in a small or in large sample, the remaining students judged whether an outcome that is *less* extreme than the specified critical value is more likely to occur in a small or in large sample. Then, Kahneman and Tversky found the following results:

| | More than 60 % (Form I) | Less than 60% (Form II) |
|---|---|---|
| The larger hospital | 12 | 9* |
| The smaller hospital | 10* | 11 |
| About the same | 28 | 25 |

(The correct responses were stared.)

Normatively[1], an extreme outcome is more likely to happen in a small sample. Therefore, in form I, the correct answer is that the smaller hospital is more likely to have the days on which more than 60% of the babies born were boys. In form II, on the other hand, such an outcome is more likely to happen in a larger sample. However, more than half of the participants in this study chose "about the same" in both forms. According to Kahneman and Tversky, student responses showed no systematic preference for the correct answer. Indeed, one can hardly explain the pattern in the student responses. Moreover, in another study, a group of college students was given a slightly different version of the problem in form I and majority of the students chose "no difference" (equivalent to "about the same") (Shaughnessy, 1977). When students were asked why they chose that particular answer, Shaughnessy reported that students evaluated the chance of getting a certain percent of boys as the same in any hospital, without any regard to the sample sizes.

**Availability Heuristic**

The availability heuristic occurs when people evaluate the likelihood of events on the basis of how easy to recall the particular instances of the event for them (Tversky & Kahneman, 1973). In these cases, people's predictions of the probabilities are often

[1] In the literature, "normative" is used to refer some theoretical model for assigning probabilities or likelihoods to events.

mediated by an assessment of availability. For instance, one may assess car accident rate in a particular location by recalling a car accident incidence he had in that location previously. To investigate such systematic biases based on the reliance on the availability heuristic, Tversky and Kahneman conducted a series of experimental studies in which students judged the frequencies or the probabilities of the events.

In one study, Tversky and Kahneman (1973) explored the role of availability in the evaluation of binomial distributions. The seventy-three 10th and 11th grade students were given the following task:

> Consider the following diagram:
>
> X X O X X X
> X X X X O X
> X O X X X X
> X X X O X X
> X X X X X O
> O X X X X X
>
> A path in this diagram is any descending line which starts at the top row, ends at the bottom row, and passes through exactly one symbol (X or O) in each row.
> What do you think is the percentage of paths which contain
> 6-X and no-O ____%
> 5-X and 1-O ____%
> .
> .
> .
> No-X and 6-O ____%
> Note that these include all possible path-types[2] and hence your estimate should add to 100%. (Tversky & Kahneman, 1973, p. 216)

Tversky and Kahneman initially postulated that the students first glance at the diagram and predict the relative frequency of each path-type on the basis of how easy it is to construct individual paths of this type. Accordingly, students would erroneously evaluate paths of 6 X's and no O to be the most frequent since there were many more X's than O's

[2] The actual distribution is binomial with p=5/6 and n=6 from a normative perspective.

in each row of the diagram. Indeed, of the 73 students, 54 responded that there were more paths consisting of 6 X’s and no O than paths consisting of 5 Xs and 1 O. In reality, however, the latter is more frequent. Hence, Tversky and Kahneman’s study (more extensive than what is discussed here) suggests that people have a tendency to make predictions based on how accessible instances of an event are to the memory, or on the relative ease of constructing particular instances of the event.

**Outcome Approach**

Konold (1991) introduced another terminology “outcome approach” on which many people make decisions about probability tasks. More specifically, Konold found that outcome-oriented students do not perceive the results of a single trial of an experiment as one of many such trials. As a result, they often simply employ the idea of “anything can happen” when they believe that no predictions can be made about random phenomena. Next, I briefly examine the outcome approach for making probability judgments as it is discussed in Konold et al. (1993).

In one study Konold and his colleagues investigated students’ probabilistic reasoning. In this study, the age and background of the subjects varied: 16 high school students who were to attend a workshop as part of Summermath program, 25 volunteer college students enrolled in a remedial-level mathematics course, and 47 undergraduate students enrolled in a statistical methods course. As part of the questionnaire, students were asked to choose from among possible sequences the “most likely” to result from flipping a coin five times in the following task:

Which of the following is the most likely result of five flips of a fair coin?

a) HHHTT
b) THHTH
c) THTTT
d) HTHTH
e) All four sequences are equally likely. (Konold et al., 1993, p. 397)

In contrast to the findings in Kahneman and Tversky (1972), only a small percentage of students seemed to use the representativeness heuristic in their reasoning and the majority of the students (72% overall) correctly responded that the sequences are equally likely to occur. Nevertheless, in a follow-up question where students were asked to select the "least likely" outcome, only about half of these students again answered that all four sequences were equally (un)likely. The results showed that this proportion was even lower among high school students. Konold and his colleagues called this pattern of response the "M-L (most-least) switch".

According to Konold et al. (1993), the students' reasoning reported in this study was consistent with the outcome approach. They argued that when asked about the most likely result, students interpreted it as to predict what would happen. Hence, they selected the answer "equally likely" because "the 50% probability associated with coin flipping suggests to them that no prediction can be made" (Konold et al., 1993, p. 399). Further, students were asked to explain their answer in the problem. When the students justified the response of "equally likely", the statements, such as "anything could happen," were interpreted as indications of the outcome approach. A few students' justifications, particularly for the question of the least likely sequence, were found consistent with the representative heuristic. For example, one student chose the sequence HHHTT as the least likely outcome because three heads in a row seemed unlikely, possibly by noting how well a sample represents the randomness of process that generates it. Thus, Konold and his colleagues conjectured that the M-L switch stemmed from a change in perspectives, from an outcome approach to the representativeness heuristic. However, this inconsistency is probably not problematic to the students since they responded on the basis of different frameworks.

**Law of Small Numbers**

The tendency to regard small samples as highly representative of the population from which they are sampled and to use them as a basis for inference and generalizations is a misapplication of the law of large numbers to small samples (Tversky & Kahneman, 1982). The belief in the law of small numbers causes someone to hold an unwarranted confidence in the validity of conclusions based on small samples and to underestimate the effect of sample size on sampling variability. Therefore, it hinders the idea that extreme outcomes are more likely to occur in small samples.

**Illicit Use of the Proportional Model**

Van Dooren et al. (2003) provided an insight into the misconception concerning the neglect of sample size reported by Fischbein and Schnarch (1997) when they elaborated on the "illusion of linearity" referred by a tendency to apply linear or proportional models improperly to any situation. For example, in Fischbein and Schnarch (1997), students from different grades ($5^{th}$ through $11^{th}$) were given a questionnaire consisting of some probability problems related to well-known probabilistic misconceptions. One of the problems was as follows:

> The likelihood of getting heads at least twice when tossing three coins is:
> smaller than
> equal to
> greater than
> the likelihood of getting heads at least 200 times out of 300 times. (ibid, p. 99)

According to the law of large numbers, as the sample size (or the number of trials) increases, the relative frequencies tend to approach the theoretical probabilities. However, the majority of the students in this study thought that the probabilities were equal and the percentage of incorrect responses got higher as the age of the students increased (from 30% of the fifth graders to 75% of the eleventh graders) . These students reasoned that 2/3 = 200/300 and thus the probabilities were the same. In this particular example, students did not pay attention to the role of sample size in calculating probabilities. Instead, they overgeneralized the linear or proportional model in comparing the probabilities of the two events. Hence, based on the findings of this problem and its variants in other studies, Van Dooren et al. (2003) asserted that certain systematic errors in probabilistic reasoning could be a result of an unwarranted application of proportionality.

**Equiprobability Bias**

Lecoutre (1992) stated that people tend to view all possible outcomes of "purely random" situations as equally likely. For instance, when two dice are simultaneously rolled, what are the chances of obtaining a total of 9 vs. a total of 11? With regard to the dice problem, Lecoutre and her colleagues observed the equiprobability bias in students' responses. In particular, these students believed that it was equally likely to get 9 and 11 "because it is a matter of chance" (p. 561). In other words, they assume that any random event is equiprobable by nature in association to the ideas of chance and luck. In the dice problem, nevertheless, there is a greater chance to obtain 9 because there are twice as many different ways to get it (i.e., 5 and 4, 4 and 5, 3 and 6, 6 and 3; vs. 5 and 6, 6 and 5).

Moreover, Pratt (2000) reported on Lecoutre's equiprobability bias in his study where a pair of students (10-year olds) assumed that the totals of two dice and of two spinners with three equal parts each labeled 1, 2, and 3 would be uniformly distributed. For example, students expected that all the totals for either two dice or two spinners were equally easy or hard to obtain. One student reasoned that the chances of getting any total were equally likely because each die was fair and the combinations of two dice must be fair as well. Moreover, in making sense of two spinners, the other student stated that there was a 50-50 chance of getting any total between 2 and 6. The misuse of the phrase "50-50 chance" was also documented by Tarr (2002). In Tarr's study, prior to instruction, many fifth-grade students incorrectly applied the phrase "50-50 chance" to (1) a situation in which all outcomes were equally likely to happen, such as a sample space including one black marble, one blue marble, one red marble, and one yellow marble in a jar, and (2) a sample space with two outcomes, such as one blue marble and four yellow marbles in a jar.

It appears that students tend to overgeneralize the equiprobable assumption of certain probability events, such as a six-sided die and a spinner with equal partitions, which are commonly used in school or in games of chance, to situations that are non-equiprobable. Even the phrase "50-50 chance" which is associated with equal chances can be misapplied to situations when there are three or more equally likely outcomes and there are only two outcomes, but which are not equally likely to occur.

## Students' Conceptions of Probability

Research on children's ideas of probability goes back to Piaget's work in which the formation of the physical aspects of the notion of chance, the basis of the quantification of probabilities, and the development of combining operations, such as combinations, permutations, and arrangements, in (four-to-twelve-year-old) children's ideas were examined (Piaget & Inhelder, 1975). In this work, Piaget and Inhelder proposed a three-stage developmental model which corresponds to Piaget's well-known developmental stages, namely pre-operational, concrete operational, and formal operational, when they investigated children's spontaneous ideas about probability using the clinical method[3] through a variety of novel tasks relevant to chance events. These tasks involved understanding of possible arrangements in a random mixture of balls and different forms of distributions of balls (uniform, centered, and skewed) in inclined rectangular boxes; understanding of the likelihood of outcomes as a ratio using a spinner, tossing a counter (a cross on one side and a circle on the other side), and drawing marbles from a sack; and understandings of combinations, permutations, and arrangements of elements, such as colored counters. In this section, I discuss the study of Piaget and Inhelder as it relates to children's understanding of forms of distributions of objects and quantification of probabilities because of their relevance to my study.

Piaget and Inhelder (1975) anticipated that students could build upon their intuitions of random mixture to reason about the fortuitous distributions. In particular, they argued that final positions of the objects in the mix and their paths to these positions

[3] The clinical method is discussed in detail in the next chapter.

would form a certain distribution in which the form of the whole would be tied into the notion of probability in children's thinking. Then, Piaget and Inhelder conducted a study with children of ages 4-12 to examine their understandings of uniform and centered distributions. They used inclined boxes with a funnel-like opening in the middle of the upper part of the box and equal-sized (2, 3, 4, or more) slots partitioned by a divider at the bottom. Children were asked to make predictions and to do experiments by dropping different numbers of balls from the funnel at the top of the box and then to explain the arrangements of the balls into the slots at the bottom. According to Piaget and Inhelder, in the first stage, young children (four to six years old) lacked an understanding of a distribution of the whole as they failed to predict or generalize the symmetrical dispersion of the balls in the slots. Although seven-to-ten-year-old children began to understand the dispersion as a whole with more or less precise symmetry, they failed to recognize the role of large numbers of balls in the second stage. For instance, there was a tendency to think that a small asymmetry of the number of balls in the slots would increase, rather than diminish, as more balls were dropped repeatedly. In the third stage, children (eleven to twelve years old) began to quantify the distributions looking at the number of balls in the slots, such as "just about equal-eight more" (p. 47). They also came to understand the role of large numbers in the regularity of distributions, such as the fortuitous differences diminish as the number of balls increases (a difference of 8 in 50 is bigger than 20 in 200 and 20-difference in 200 is bigger than 20 out of one thousand).

With regard to the quantification of probabilities, Piaget and Inhelder claimed that children's development of comparing the likelihoods of events depended on their ability to relate the part (favorable outcomes) with the whole (all possible outcomes) and their understandings of the outcome space and the combinatoric operations. For instance, Piaget and Inhelder asked children which outcome would most likely to happen if they were to draw all the marbles two at a time from a sack in which there were equal number of blue and red marbles. In stage one, children tended to believe that picking the same-color pairs (BB or RR) would be more likely than getting the mixed pairs (BR or RB) and some did not even think of a possibility of a mixed pair. When children initially noticed the possibility of the mixed case (RB or BR) in the second stage, they quickly responded that the mixed pairs would come out more frequently than the pairs BB or RR, but failed to quantify the frequency of the mixed outcomes. In stage three, children formulated numerical quantification of possible outcomes (i.e., getting RB or BR is twice more likely than getting either BB or RR) on the basis of their understandings of possible outcomes and combinatoric reasoning skills.

In general, Piaget and Inhelder viewed the notion of ratio and proportion based on the combinatoric operations (combinations, permutations, and arrangements) as the origin of the development of chance. They also primarily focused on “the acquisition of a complete (or more advanced) theoretical concept of chance by children” (Van Dooren, et al., 2003, p. 116). However, there is another major contribution by Fischbein (1975) to the research on the formation of the probability concept. According to Fischbein’s perspective, intuitions that are global and immediate in nature are considered central to the children’s development of the concept of probability and that these probabilistic intuitions can be derived from the individual experiences as well as from the formal education in school. Thus, Fischbein (1975) claimed that the development of conceptions of probability could be mediated through instructional intervention and social interactions.

To examine the role of intuitions in the development of probabilistic thinking, Fischbein and his colleagues conducted a series of studies. In a relatively recent study, Fischbein, Nello, and Marino (1991) investigated the origins and the nature of some probabilistic intuitions regarding the types of events (i.e., impossible, possible, and certain events), the roles of different embodiments of identical mathematical or stochastic structure, and the compound events. The participants of this study included 211 elementary school students (ages of 9-11), 278 junior high school students (ages of 11-14) without prior instruction on probability, and 130 junior high school students (ages of 11-14) with prior instruction in probability. In the usual classroom setting, the students were asked to solve fourteen probability problems with an explanation.

Fischbein and his colleagues found that children did not necessarily have a spontaneous understanding of the concepts "possible, impossible, and certain." Some students seemed to have difficulty in referring to certain events when they tended to decompose the certain totality into a number of possibilities. For instance, a student thought that obtaining a number smaller than 7 in rolling a die was "possible" because one might get any number smaller than 7. Moreover, the concept of impossible seemed to be identified with either "rare" (i.e., "It is impossible because the probability is very small") or "uncertain" (i.e., "It is impossible because one cannot be sure").

In this study, students also were asked to compare two different situations with the same mathematical (stochastic) structure, such as the probability of obtaining three 5s either by rolling a die three times or by rolling three dice simultaneously. Fischbein et al.

found that the majority of the students recognized the equal probabilities as they got older and received some instruction on probability. However, many students thought that there was a higher chance of getting 5, 5, 5 by rolling a die three times because the process could be better controlled to get the certain outcome, such as by using the same kind of roll each time.

According to Fischbein et al. (1991), an intuitive understanding of the probability of a compound event in relation to the magnitude of the sample space developed spontaneously with age. For instance, in rolling two dice, the probability of getting the sum of 3 is the same as the probability of obtaining the sum of 11 because 3 can be obtained with rolling 5 and 6 or 6 and 5 and similarly one can get 11 by rolling 2 and 1 or 1 and 2 (hence there are two possible ways to get both sum). However, the researchers pointed out several difficulties that conflicted with the correct evaluation of the probabilities in compound events. Firstly, children did not have an intuition to consider order of the outcomes, such as (5, 6) and (6, 5) in rolling two dice. Secondly, students tended to disregard the specific limitation in the chance experiment. For example, numbers like 8, 10, and so on, were also considered in two-dice game when students listed the possibilities of getting the sum of 11. Thirdly, many students lacked a systematic way for generating all possible outcomes related to a probability event. Fourthly, due to the availability heuristic (discussed earlier in this chapter), students tended to compare correctly the possibilities of getting 2 or 12 than obtaining 3 or 11 when rolling two dice (i.e., easier to identify 1+1 or 6+6 than 2+1 and 1+2 or 5+6 and 6+5). Finally, children and even adolescents seemed to estimate equal probabilities based on the misunderstanding of the notion of chance. For example, students believed that any two chance events had equal probabilities because the outcome depended only on chance no matter what the given conditions were.

To summarize, for Piaget and Inhelder (1975), the development of probability concept depended on children's ability to recognize the relationship between the part and the whole with regard to the outcomes of a random event, and their conceptions of sample space, combinations, and permutations. For Fischbein and his colleagues (Fischbein, 1975; Fischbein et al., 1991), children's intuitions of probabilistic concepts were of importance as their development into more formal concepts of probability could be mediated through systematic instruction and experiences based on social interactions. Next, I focus on a set of recent studies that documented a detailed model of students' probabilistic reasoning by considering their performance along several aspects of normative reasoning in stochastic theory (e.g., Horvath & Lehrer, 1998; Vahey, 1997; and Vahey et al., 2000).

## Models of Students' Probabilistic Reasoning

Horvath and Lehrer (1998) identified five distinct, but related, components of the classical model of probability that were used to investigate the understanding of chance and uncertainty: (1) the distinction between certainty and uncertainty (predictable and unpredictable nature of certain outcomes), (2) the nature of an experimental trial (one's determination of whether the probability events, such as two spins of a spinner, are identical), (3) the relationship between individual outcomes and patterns of outcomes (individual events may show unpredictable outcomes, whereas distributions of these outcomes often yield predictable patterns), (4) the structure of events (how the sample space relates to outcomes), and (5) the treatment of residuals (difference between predicted and actual results). In the $2^{nd}$ grade classroom study, students carried out experiments with one or two 6-sided and 8-sided dice in pairs. They were asked to first predict, then to generate results and to justify the relationship between expected and obtained outcomes. They used bar graphs to record their results and sample spaces.

Next, I summarize the findings of Horvath and Lehrer (1998) in relation to the components of the classical model of probabilistic reasoning listed above:

*The distinction between certainty and uncertainty*. Many children initially assumed that rolling dice was not completely random. Until beginning to collect data from dice rolling, they expected that they could predict the next outcome in rolling dice, such as lucky numbers.

*The nature of an experimental trial*. Half of the students believed that the way of tossing the dice will affect the outcome while the rest disagreed. The dispute was resolved when they all agreed to roll dice out of a cup to ensure the uncertainty of outcome.

*The relationship between individual outcomes and patterns of outcomes*. As they had more experience with the tasks, most students believed that global patterns of events were more predictable than local outcomes (any single outcome). They tended to make predictions about the distribution of outcomes in long run, such as 50 trials, based on the past results generated after 20 trials. Students' reasoning about the predictability of the distribution of outcomes and uncertainty of a single outcome was also affected by the discussion of the sample space in this particular task. Students became highly confident that they could predict the shape of the distribution of results rolling a die many times.

They, however, were not very convinced that they could predict the result of any single roll. Moreover, the graphical representations highlighted the general shape of the distribution and significance of results when students looked at the sample space graphs (based on both combinations and permutations) and the actual outcome graph. It also encouraged students to notice relationships at a global level (i.e. distributions) rather than at a local level (i.e. comparing individual outcomes).

*The structure of events*. In discussing the relationship between the sample space and outcomes, when generating all the possible ways to get the totals of rolling two eight-sided dice, students debated over whether order was important (i.e., 1,5 and 5,1 are two different ways for getting 6-permutations) or not. Some students argued that various combinations, such as 1,5 and 5,1, were equivalent because of the commutative property of addition (i.e., 1+5 = 5+1). Students' understanding of the relationships between the number of possible outcomes and the distribution of actual outcomes was quite weak. When they made predictions about the distribution of results, only a few of them could use the number of possible outcomes with no support of discussion and the bar graphs in front of them.

*The treatment of residuals*. A few students came to understand residuals in terms of the relationship between outcomes and the sample space, such as its overall (triangular) shape based on the permutations for each outcome (see Figure 2). When the results of the experiment (i.e., the frequency of each outcome after the experiment) did not exactly match the predicted outcomes, students tended to reason by their lucky numbers or by their past experiences with games involving dice. Moreover, after some experiences with noticing residuals, most students seemed to make generalization based on the previous results and change their predictions accordingly.

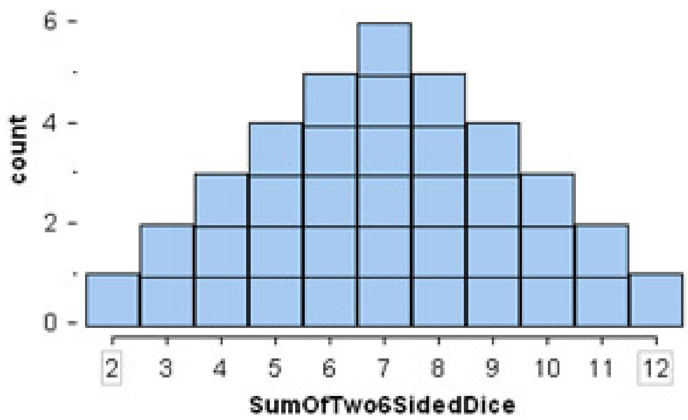


Figure 2. Graph for the sample space of the sum of two 6-sided dice.

Similarly, Vahey and his colleagues (Vahey, 1997; Vahey et al., 2000) noted four aspects of normative reasoning in probability situations: (1) randomness (understanding that the game is based on non-deterministic mechanism), (2) outcome space (enumerating all possible outcomes), (3) probability distribution (probability of each outcome), and (4) validity of evidence (the law of small or large numbers). Vahey and his colleagues acknowledged the notion of fairness as a motivating and productive area of inquiry for students investigating probability in computer-based activities. The researchers used the four interrelated conceptual areas of probability theory to examine the seventh-grade students' probabilistic reasoning during their interactions with the Probability Inquiry Environment (PIE), a collaborative, guided-inquiry learning environment, in which students were asked to evaluate the fairness of games of chance.

In Vahey (1997), pairs of seventh-grade students were given tasks related to coin flipping. For each game, students were asked to evaluate the fairness of the games based on the information given in the PIE software interface including questions, a probability tree (see Figure 3), and a bar chart, by predicting, playing, and interpreting the results. It was found that initially students had different conceptions of fairness. Those included equal chances of winning, some variation which is not systematically favorable to one team, no cheating, and possibility to win for each team.

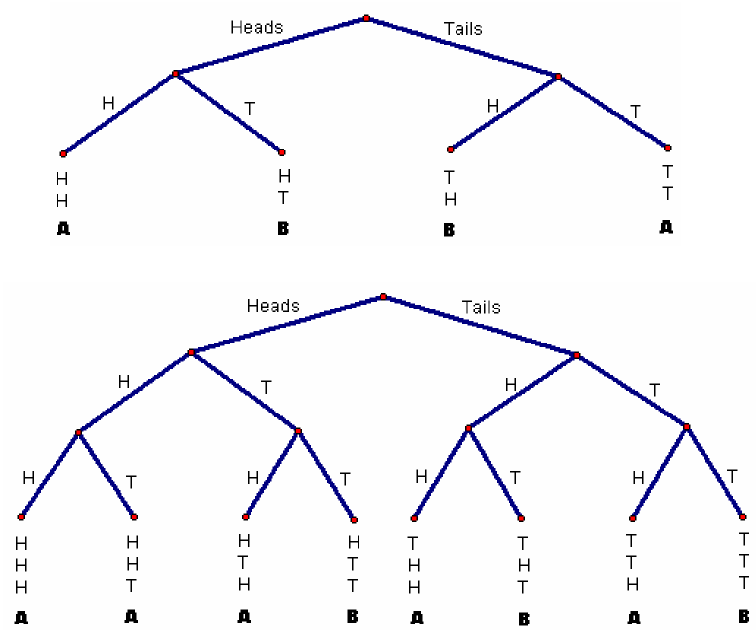


Figure 3. Probability trees for Two-Penny and Three-Penny games showing the possible outcomes for each team to win.

Vahey (1997) argued that a wide variety of ideas students had, either consistent or conflicted with the normative reasoning, made it difficult to characterize students' reasoning only on specific misconceptions. Therefore, the researchers considered ways of eliciting students' different ideas based on the four aspects of the normative reasoning in probability:

*The randomness*. Students believed that the outcomes of coin flip were non-deterministic. Their explanations differed though. While some students stated a notion of randomness as being based on "luck and chance," or that the results between trials might vary, some of them viewed randomness as nothing could be predicted about future events or anything could happen (i.e., Konold's outcome approach).

*The outcome space and the probability distribution*. Students had difficulty in making a distinction between the individual outcomes and the combination of all outcomes that could score a point for a team. For instance, they reasoned that it was easier to get HHT than getting HHH because there are more HHT. Moreover, one student believed that some outcomes, like TTT or THT, were less likely to occur since it was "too much of a pattern" (Vahey, 1997, p. 11). Vahey argued that students would often switch between talking about the number of possible outcomes for each team to win and the probabilities of outcomes. Like in Horvath and Lehrer (1998), students rarely made reference to the outcome (or sample) space on their reasoning.

*The validity of evidence*. Students' belief on the law of small numbers appeared in two different ways. In the data-driven case, students tended to give up their theory based on small number of trials (i.e., 10 or less), or not to generate a theory in the absence of data. In the theory-driven case, students seemed not to accept data as relevant when the data were in conflict with their theory.

## Reasoning about Distributions in Data Analysis

A number of researchers focused on the notion of distribution as a big idea in statistics and examined students' reasoning about distributions within statistical reasoning and modeling data. For example, Cobb and his colleagues (Cobb, 1999) approached statistical reasoning in such a way that seventh-grade students were able to reason about distributions with strong connections among various statistical topics, including the statistical measures and the representations. These researchers believed that students could develop their own understanding of central statistical ideas and topics as they engaged with data analysis through statistical reasoning. For instance, when students described given a set of data in terms of its qualitative features, the students began to reason using trends and patterns from the context of the problem.

Cobb and his colleagues also focused on distributions in order to bring the topics, such as mean, median, mode, spread-out-ness, and graphical inscriptions, together so that students could develop understandings of these topics either by organizing distribution or describing its characteristics through statistical reasoning. Therefore, they designed data sets for analyses in which students could make decisions in the context of a real-life problem situation. Moreover, students were encouraged to justify their reasoning in the whole-class discussions to develop their own statistical understandings through these tasks.

In their approach, Cobb and his colleagues (Cobb, 1999) pointed out that students' understanding of the distribution would be essential to reason about data as aggregates. It was argued that students should think about data sets as entities that are distributed within a space of possible values rather than as a collection of individual data points (McClain et al., 2000). Therefore, the researchers emphasized the exploration of qualitative characteristics of collections of data points (Cobb, 1999). Moreover, these characteristics were treated as features of the situation from which the data were generated. Thus, it was necessary to investigate how students understood the aspects of distributions and how they interpreted a distribution as they reasoned statistically.

Since a graphical representation of the data was visual, the shape of the data initially helped students to understand and interpret the given data in a context. Students began to use informal language to describe the shape of the distribution. For example, such words as clumps, clusters, bumps, gaps, holes, spread out, and bunched together, were used to describe where most of the data were, where there were no data, and where there were isolated pieces or natural subgroups of data. The researchers found that this was an important step toward exploring the qualitative characteristics of a distribution.

Lehrer and Schauble (2000) were specifically interested in fostering the emergence of model-based reasoning in mathematics and science in the context of data investigation in the elementary grades ($1^{st}$-$5^{th}$). The researchers have seen the data modeling as an important resource for exploring the world. In their work, data modeling served "as part of a chain of inquiry, encompassing question-posing, the construction of attributes, and the creation of structures and displays that function to aggregate attributes over multiple cases" (p. 130). Similar to Cobb and others' study (Cobb, 1999), Lehrer and Schauble heavily relied on reasoning about aggregates (e.g. distributions) and acknowledged that this form of reasoning appeared to be mastered over an extended period of time and depended on thoughtful instructional support and opportunities for practice. The analysis of classroom episodes (Lehrer & Schauble, 2000) suggested that older students were able to consider the properties of distributions, such as measures of center and dispersion. Accordingly, these students used them as resources to investigate problems that required simultaneous consideration of center and variability.

## Key Aspects of Literature

In this chapter, I discussed the existing knowledge base regarding the reasoning about chance and data that formed a basis upon which this study was conducted. In this section, I describe the key aspects of the existing literature that I used to frame my study of fourth-grade students' reasoning about distributions in chance events.

Firstly, the historical development of the probability concept provides insights into the different interpretations of probability. Earlier in this chapter, it was discussed that the distinction between the *scientia* (knowledge demonstrated deductively from first principles) and the *opinio* (beliefs testified by some authority or through God-given signs), and the transformation of what constituted acceptable evidence gave rise to the dual meaning of probability. Hence, the concept of probability was historically used to both describe the degrees of belief relative to our background knowledge (the epistemic notion) and refer to the tendency of certain random events, such as flipping a coin, to generate the stable regularities concerning the physical and geometric properties of the chance event by computing the relative frequencies in the long run (the statistical notion). This duality in effect recognizes both formal and informal uses of probability that can be encountered in children's reasoning about uncertain events. Also, it is of importance to note the different views of probability, such as the classical, frequentist, and subjective, that can be used to recognize various beliefs about probability held by students. More specifically, the classical probability refers to the ratio of the number of favorable cases in an event to the total number of equally likely cases. Then, the frequentist probability of an event is defined as the limit of its relative frequency in a large number of trials. Lastly, the subjective probability is considered as the degree of belief, like the *opinio*.

Several studies documented persistent erroneous conceptions (or misconceptions) and strategies students held and employed in judging the likelihood of uncertain events. Those included the representativeness and availability heuristics, the outcome approach, the law of small numbers, the illicit use of the proportional model, and the equiprobability bias. These misconceptions and errors in probabilistic reasoning is of interest to this study because students come to the classroom with prior conceptions and beliefs about physical, deterministic, and probabilistic phenomena based on personal experiences and thus they may fall prey to some of these misconceptions. Accordingly, young students may have difficulties in building intuitions for probability in chance events and developing the normative probabilistic reasoning.

Moreover, the work on children's spontaneous development of probability concepts, including the formation of the physical aspect of chance, the role of large numbers, the basis of quantifying probability, and the development of combinatoric operations, and the studies on children's probabilistic intuitions provided a foundation for formulating a conceptual trajectory in this study. The research suggests that children's ideas of probability concept develop in relation to the formation of ratio and proportional reasoning, and combinatoric operations. In addition, children's intuitive understanding of probability can be mediated through instruction before they develop strong understandings of part-whole and combinatoric skills.

Characterizing students' probabilistic reasoning only on particular misconceptions does not provide a complete explanation of a variety of ideas students already possess. Thus, some researchers considered several aspects of normative stochastic reasoning in documenting the models of students' reasoning. For example, they identified randomness, the distinction between certainty and uncertainty, the nature of the experimental trials, outcome space and probability distribution, the relationship between individual outcomes and patterns of outcomes, the treatment of difference between the expected and actual outcomes, and the validity of evidence as the characteristics of probabilistic reasoning. Then, the students' intuitive reasoning was compared to the normative reasoning along these aspects. A variety of students' ideas elicited through this framework guided this study in determining what probability concepts to focus on, how to structure and sequence the tasks, and how to analyze and interpret students' spontaneous probabilistic understanding and reasoning.

Finally, the research on students' reasoning about distributions in data analysis provided the foundation of the focus of the study on the distributions in chance events. Therefore, in this study, the notion of distribution as the aggregates of data is emphasized. Moreover, students' informal understandings of the qualitative aspects of distribution in relation to the middle, spread or variability, and shape were of interest to this study as these conceptions can be developed into more formal understanding of distribution in chance events, such as the quantification of the most likely and the least likely outcomes.

## Purpose and Research Questions

As discussed above, the review of the literature provides a knowledge base on students' ideas about probability in a variety of ways. Moreover, the research on students' statistical reasoning about distributions in data analysis (e.g., Cobb, 1999; Lehrer & Schauble, 2000) led me to consider the notion of distribution in the context of chance in bringing the discussions of probability related topics together (i.e., sample space[4], combinations, permutations, probability, the law of large numbers, representations, and inscriptions etc.). I also sought to understand the role of the notion of distribution as a link between data and chance which is apparently absent in the discussions of probability and statistics in research and instruction. Hence, the purpose of this design study was to characterize a conceptual corridor that contains the conceptual trajectories taken by the fourth-grade students based on their conceptions of probability and reasoning about distributions in chance events by the design of a sequence of tasks.

Consequently, as stated in Chapter 1, the main research question was:

- How do students develop reasoning about distributions when engaging in explorations of chance situations through a sequence of tasks, in which students were asked to provide predictions and explanations during the experiments and simulations with objects, physical apparatus, and computer environment?

To understand this larger question, I developed four supporting research questions based on the review of the literature:

- What are the students' prior knowledge about probability concepts and probabilistic reasoning?
- What kinds of informal knowledge and strategies can serve as starting points?
- What are the conceptual trajectories that students take during the teaching experiment?

[4] See the definitions of these terms in Appendix E.

- What are the resources (learned ideas) that students bring into the understanding of probabilistic concepts and reasoning?

# CHAPTER 3

# THEORY AND METHODOLOGY

The review of literature in the previous chapter outlines the knowledge bases for students' ideas about chance and their reasoning under uncertainty. Methodologically, many of these studies (e.g., Fischbein & Schnarch, 1997; Kahneman & Tversky, 1972; Lecoutre, 1992) used questionnaires to assess students' understanding and reasoning. In particular, many of the tasks involved multiple-choice items. Although student responses to the specific survey items in these studies permitted researchers to identify various misconceptions, I agreed with Shaughnessy (1992) that clinical methodologies, such as clinical interviews and teaching experiments, provide an opportunity to elicit a wider variety of student ideas concerning stochastic tasks. Therefore, in this study, I conducted pre-and post-clinical interviews and a small-group design experiment to examine students' understandings of probability and their reasoning about distributions in the probability situations.

In designing the study and creating the tasks, I have mainly built from the constructivist and the socio-cultural traditions which are discussed next. In the subsequent sections, I describe the design of the study, the participants, the pilot study, the study instruments and tasks, the methods of data collection, and the data analysis procedures.

## Theoretical Framework

This study employs a small-group design experiment model. According to Cobb, Confrey, diSessa, Lehrer, and Shauble (2002),

> Prototypically, design experiments entail both “engineering” particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. This designed context is subject to test and revision, and successive iterations that result play a role similar to that of systematic variation in experiment. (p. 9)

In particular, this study aims to investigate students’ reasoning about distributions in chance events within tasks designed to support the development of students’ thinking. Moreover, design experiments involve identifying “conjectured starting points, elements of a trajectory, and prospective end points” as well as generating “stable conjectures about both significant shifts in student reasoning and the specific means of supporting those shifts” (ibid, p. 11) (see also Confrey & Lachance, 2000). The main focus of design experiments, in particular one-on-one and classroom teaching experiments, is to develop theories of the learning of individual students or of a classroom community, and ways of supporting their learning of domain specific content. The study of intellectual development of children can be a major source to generate those theories.

In the chapter tracing the historical development of design experiments, Confrey (2006) argues that the purpose of a design experiment is to articulate two related ideas, namely *a conceptual trajectory* and *a conceptual corridor*. In the context of a design experiment, a conceptual trajectory is a possible pathway that students can navigate during any particular set of instructional episodes and a conceptual corridor is a broader concept that refers to all likely conceptual trajectories (Confrey, 2006). According to Confrey, in constructing the conceptual corridor, one needs to start with students’ prior knowledge and an initial problematic (the use of the term “problematic” in Confrey (1998) refers to students’ interpretation of the problem in relation to his or her purposes, goals, expectations, and prior knowledge). Moreover, the borders of the corridor are determined by the design of a sequence of tasks. Confrey (2006) also defines *landmarks* within the corridor, which are student conceptions that can be anticipated initially and identified based on student responses and formative assessments. Furthermore, it is noted that the specification of such a conceptual corridor should be flexible enough for students to progress through that space.

The teaching experiment in the current study considered students' conceptions and reasoning, and some tools to support their development along possible conceptual trajectories in a conceptual corridor. In doing so, the work of two theorists, Jean Piaget and Lev Vygotsky, is of particular importance because of their contributions to the study of children's thinking and reasoning in mathematics and science education. Next, I will present two major traditions, constructivist and socio-cultural perspectives, that are drawn upon the theories of these scholars in mathematics education research. Also, the methodological implications from these perspectives will be discussed.

## Constructivism

### The Relation between Knowledge and Reality in Constructivism

According to the constructivist view--an interpretation of Piaget--children's minds are not blank slates. Rather, students' alternative conceptions must be taken into account, using significant discussions and interactions around their various strategies (Confrey, 1994). Glasersfeld (1995) stated the two basic principles of constructivism: (1) "Knowledge is not passively received but built up by the cognizing subject" and (2) "The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (p.18). While Glasersfeld called a stance that accepts only the first principle “trivial constructivism”, he called a stance that accepts both principles simultaneously “radical constructivism.” Hence, radical constructivism[1] suggests that the individual builds up the knowledge, rather than passively acquires it. Moreover, in the radical constructivist view, the relation of reality and knowledge is seen as “an adaptation in the functional sense,” which refers to organization of our knowledge of the world constituted by our perceptions and experiences, rather than “a more or less picture-like correspondence or match” (Glasersfeld, 1984, p. 20).

Further, from a constructivist perspective, we do not have access to an objective reality, in the sense that there is a reality independent of our way of knowing (Simon, 1995). Hence, Simon notes that we can never know whether a concept matches an objective reality. In relation to the notion of fit (rather than match) Glasersfeld (1995) used the concept of *viability* to indicate whether something fits in our experiential world saying that “actions, concepts, and conceptual operations are viable if they fit the purposive or descriptive contexts in which we use them” (p.14).

**The Construction of Knowledge and Piaget’s Scheme Theory**

In his book *Genetic Epistemology* (1970), Piaget explained knowledge construction, particularly in relation to the logical and mathematical knowledge. He said that knowing something does not mean copying it; rather it means acting upon it. Piaget distinguished physical and logical mathematical knowledge. While the former refers to the knowledge based on experiences in general, the latter applies to structures that are abstract. According to Piaget, there are two types of abstraction: 1) simple abstraction

[1] Through out this paper, “constructivism” will refer to “radical constructivism” as described by Glasersfeld here.

which is derived from the object that we act upon, and 2) reflective abstraction which is drawn from the action itself. The second one is reflective in a sense that "at the level of thought a reorganization takes place" (ibid, p. 18). Furthermore, reflective abstraction always involves coordinated actions in mental processes.

Piaget (1970) considered coordinated actions as the basis of logical mathematical thoughts. For him, coordinated actions become mental operations and a system of these operations constitute a structure. Piaget's notion of *scheme* refers to "whatever is repeatable and generalizable in an action" (ibid, p. 42). Hence, scheme theory functions to explain the stability and predictability of actions (Confrey, 1994). To interpret Piaget's scheme theory, one needs to understand two fundamental processes: *assimilation* and *accommodation*. According to Piaget, assimilation involves an incorporation of new experiences and perceptions of the world to the existing schema whereas accommodation refers to adaptation of the existing schema to a new structure. Moreover, these two processes work in a dialectical relationship. When a scheme leads to a perturbation, the problematic will be called to action. Then, accommodation takes place in order to maintain or reestablish the equilibrium.

## Socio-cultural Perspectives

### The Three Themes in Vygotsky's Theoretical Framework

A socio-cultural approach to intellectual development in Vygotsky's work is based in the assumption that human action is mediated and it can never be separated from its specific socio-cultural context (Werstch, 1991). In Wertsch's interpretation of Vygotsky's theoretical approach, there are three fundamental themes (Werscth, 1985):

(1) *The genetic, or developmental, analysis*: The claim is that human mental functioning can be understood only if we understand how and where the development has occurred. Therefore, Vygotsky's *genetic analysis* involves investigating the origins of the mental processes and the ways that they are carried out.

(2) *The social origins of higher mental functioning in the individual*: For Vygotsky, higher mental functioning in the individual emerges from the social context. Thus, the direction of intellectual development is from social to individual. This idea is stated in Vygotsky's *general genetic law of cultural development.* According to this law, there are two planes, social and psychological, on which the child's cultural development appears. More specifically, any mental function occurs first in the social plane (between individuals as an interpsychological category) and then in the psychological plane (within the individual as an intrapsychological category).

(3) *The semiotic analysis*: Vygotsky argued that higher mental processes are mediated by *tools*[2] and *signs*[3]. Development takes place when these different forms of mediation create a transformation in mental functioning. In a Vygotskian approach, mediational means are essentially social, not individual. They are the products of sociocultural history, rather than inventions of each individual or discoveries of the individual's independent interaction with nature. Moreover, individuals have access to the mediational means as part of a socio-cultural context, from which individuals "appropriate[4]" them. For Vygotsky, the mediational means play an important role in influencing others, and only later function to influence the individual.

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[2]" Tools" here refers to technical tools, such as a calculator, a graphic paper, etc.
[3] "Signs" here refers to psychological tools, such as language, algebraic symbols, and so on.
[4] Wertsch (1985) notes that this term comes from the later work of A. N. Leont'ev (1959).

**The Concepts of Internalization and the Zone of Proximal Development**

According to Vygotsky, there is an inherent connection between the interpsychological plane and the intrapsychological plane. He was particularly concerned with the processes in which mental functions are transformed from an external plane to an internal plane. Since Vygotsky viewed higher mental functions as social processes, for him the notion of *internalization* developed as “a process involved in the transformation of social phenomena into psychological phenomena” (Wertsch, 1985, p. 63). While the external processes are necessarily socially interactional, internal processes reflect certain aspects of this social interaction. Indeed, the higher mental functions derive from the mastery of external sign forms. Wertsch also pointed out that for Vygotsky, internalization is not a process of copying external reality on a preexisting internal plane; instead, it is the process by which the internal plane is formed.

Vygotsky further described his concrete ideas about the connection between interpsychological and intrapsychological functioning in relation to the concept of *the zone of proximal development* (ZPD) (Wertsch, 1985). In an attempt to explain how to evaluate interpsychological processes and relate them to their intrapsychological outcomes, Vygotsky introduced the notion of the ZPD (Vygotsky, 1978):

> [The ZPD] is the difference between [a child’s] actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (p. 86)

The practical implications of this concept are of importance. Vygotsky argued that intellectual development occurs within the ZPD of the child. He pointed out that “learning which is oriented toward developmental levels that have already been reached is ineffective from the view point of the child's overall development. It does not aim for a new stage of the developmental process but rather lags behind this process" (ibid, p. 89). From this point of view, instruction should aim at a developmental level that is just above the student's current developmental level. Moreover, in his formulation of the ZPD, Vygotsky pointed to the role of *imitation* in learning. He argued that “children can imitate a variety of actions that go well beyond the limits of their own capacities. Using imitation, children are capable of doing much more in collective activity or under the guidance of adults” (ibid, p. 88). For Vygotsky, with assistance of an adult or a more capable peer every child can do more than he/she can by him/herself.

## Methodological Implications

In order to link these philosophical traditions together, I have conducted a design study situated in a constructivist tradition informed by a socio-cultural perspective (Confrey & Kazak, 2006). Drawing on the work of Confrey (2006), one can link the work of constructivism and socio-cultural perspectives into an approach that seeks to document the development in student thinking about the relationships of probability and data analysis.

From a constructivist position, it is essential to use methods that attempt to describe the distinctive nature of the child’s thought. The two common methods that have been widely used by constructivist researchers in mathematics education to explore the thought processes of students are Piaget’s clinical method and the teaching experiment method. Hence, I used the clinical method to interview individual students before and after the study, and conducted a small-group teaching experiment to investigate students’ probabilistic reasoning about distributions. In socio-cultural research, Vygotsky's work provides insightful ideas about intellectual development, and yet the methodological implications of these ideas for empirical research are not often clear (Wertsch & Kazak, in preparation). Therefore, I discuss next only the methods that have roots in constructivism, and then I incorporate Vygotsky's ideas into my discussion of the design study in the data analysis section.

**Clinical Method**

The clinical method (or the clinical interview) originates with Piaget's earlier investigations of reasoning in children. In the method of clinical examination, Piaget (1976) combined the method of standardized testing and natural observation without falling into the pitfalls of these methods, by allowing the child to express his/her thoughts and observing the child's spontaneous behaviors or interactions. In Piaget's clinical method, the experimental aspect of the examination involves setting up a problem, making hypotheses, adapting the conditions to them, and controlling each hypothesis by testing it against the reactions that the interviewer stimulates in conversation. The clinical examination is also dependent on direct observation, which allows the researcher to capture the whole of the mental context.

Opper (1977) described the clinical method as "a diagnostic tool applied to reasoning in children. It takes the form of a dialogue or conversation held in an individual session between an adult, the interviewer, and a child, the subject of study" (p. 92). At the core of the clinical method is the hypothesis-testing feature of the method, which allows the interviewer to draw inferences about the child's thinking process by observing his or her performance on certain tasks during the interview session. In

Piagetian studies, for example, the child is presented an experiment that involves both "a concrete situation with objects placed in front of the child and a verbally presented problem related to this situation" (ibid., p. 92). The interviewer has initial guiding hypotheses to start with about the types of thinking that the child will engage in through the given tasks in which physical or spatial manipulations are acted upon the materials. In these tasks, the child is asked to provide predictions, observations, and explanations for the results of the manipulations performed on the concrete objects. On the basis of these verbal responses and actions, the interviewer tests his or her initial hypotheses. Probing questions also further clarifies these responses. Reformulation of the original hypotheses, if needed, and the formation of new hypotheses occur until the interviewer is satisfied with the exploration of the child's thinking as far as possible.

The description of the clinical method given above indicates that the interviewer has complete freedom in conducting the interviews, such as the choice of the ways to study the concept of interest and the flexibility to modify the experimental situation for each particular child (Opper, 1977). In practice, however, Opper argued that a systematic approach to conducting interviews would be needed for comparable results. Thus, a partially standardized version of this method is commonly used in research based in the Piagetian tradition. More specifically, a standard task with certain identical manipulations is presented to each participant. In relation to the task at hand, all participants are asked a number of identical questions. At the same time, the interviewer has the freedom of rephrasing the questions according to the child's responses and following up the child's novel responses and actions with additional questions or items in order to understand better the child's thinking and reasoning. Therefore, this partially standardized version of the clinical method is “an attempt to combine the more structured approach of standardized testing with the flexibility of the clinical method” (ibid., p. 95).

**The Constructivist Teaching Experiment**

The constructivist teaching experiment methodology is designed to investigate students’ mathematics (which refers to children’s mathematical knowledge), and mathematics of students (which represents the researcher’s interpretations or models of students’ mathematics) in the context of mathematics teaching (Steffe & Thompson, 2000). Steffe and Thompson use of the phrase “students’ mathematics” is based on the assumption of students’ mathematical reality as being different from the researcher’s mathematical reality. Therefore, a goal of the researcher in a teaching experiment is to construct models of students’ mathematics. The legitimacy of these models depends on the extent to which the researcher can find rational grounds for students’ language and actions.

To this end, the exploratory framework of the teaching experiment that is drawn from Piaget’s clinical method involves basically “formulating and testing hypotheses about various aspects of the child’s goal-directed mathematical activity in order to learn what the child’s mathematical knowledge might be like” (Steffe, 1991, pp. 177). During the teaching episodes, and between the teaching episodes as well, the teacher-researcher both generates and tests hypotheses about possible meanings of the students’ mathematical language and actions. However, Steffe argues that the teaching experiment is more than a clinical interview, because developing ways and means of influencing students’ knowledge is part of the experiment. Hence, in this section I will primarily focus on these distinct features of the teaching experiment methodology in mathematics education.

The researcher's role as teacher is a distinguishing characteristic of this method (Steffe, 1991). Beyond acting as teacher, the teacher-researcher also attempts to understand students' mathematics in their ways and means of thinking and reasoning. In doing so, the teacher-researcher creates situations and ways of interacting with the students that foster students' learning. More specifically, learning occurs as a result of students' interactions in particular situations that help students alter their current schemes. The researcher is interested in the assimilating schemes of students in these situations, situations which ought to be "interesting and challenging for students" (Steffe & Thompson, 2000, p. 289). According to Steffe and Thompson (2000), deliberate variation in the situations with respect to their context, material, and scope is an important part of the framework of the teaching experiment. In doing so, however, the teacher-researcher should not go so far beyond students' current schemes that it requires students to make major accommodations in their schemes. Rather, in fostering accommodation, the teacher-researcher generates situations of learning from which a perturbation results, so that students experience a moderate reorganization of their existing schemes, for example by using them in novel ways.

## Design of the Study

In order to answer the research questions set forth in Chapter 2, the work of Piaget and Vygotsky and the methodological implications discussed above convinced me to choose a design in which I could define a conceptual corridor of possible opportunities, landmark conceptions, and obstacles. To do so, a sequence of tasks, some of which were adapted from other studies, was developed based on my initial conjecture about linking the discussions of probability and data analysis through the notion of probability distribution, in light of the existing literature on students' understanding of chance and data, the historical development of the probability concept, and my mathematical content knowledge about probability and statistics. Then, a pilot study was conducted to field test the tasks developed for use in the teaching episodes. From the pilot study data, conjectures about a conceptual corridor were also developed. Based on the concepts, tools, and ideas relevant to the initial conjectures, the interview questions were selected from the literature and from the National Assessment of Educational Progress (NAEP) items (see Appendix A). Follow-up probes were used during particular encounters in the interview tasks. The individual interview sessions took place several days before and after the teaching sessions, to examine individual participants' understanding of probability concepts and their probabilistic reasoning. The teaching sessions were conducted over a period of seven weeks in which participants in groups of three completed the tasks designed for this study.

## Participants

The participants were six volunteered fourth-grade students (9-year olds), three boys and three girls (Alex, Alicia, Caleb, Emily, Josh, and Maya; all names are pseudonyms). The sample was selected as a convenience sample. The participating students were recruited through their classroom teacher at a local elementary school at the beginning of the fall of 2005. The same teacher also helped me to recruit students from her class for my pilot study in the spring of 2005. During the study, the interview and teaching experiment sessions occurred in a room (previously used for Spanish lab) at the school during the mathematics class period.

## Pilot Study

In the spring of 2005, a pilot study was conducted to refine the tasks to be used in the actual study and to develop initial conjectures about the conceptual corridor (see the detailed discussion in Chapter 4). Four children (9-year-olds; Brad, Jim, Kate, and Tana; all names are pseudonyms) participated in the pilot study, which included one-on-one interviews on the distributions task, and three teaching episodes, each about an hour long. The tasks used in the pilot study included *Distributions in Different Settings*, *Dropping Chips Experiment*, *Split-box*, *Flipping a Coin*, and *Hopping Rabbits* (see Appendix B). In these tasks, students were asked to predict, generate, represent, and interpret the outcomes of the experiments and simulate experiments with chance devices.

The video records and transcribed data from the interviews and the teaching episodes were analyzed to further refine and develop appropriate tasks for use in the actual study. Moreover, the pilot study data allowed me to develop initial conjectures about students' reasoning about distributions in probability situations as well as the means to support those conjectures. I discuss these results in Chapter 4.

**Study Instruments and Tasks**

The design experiment study involved both one-on-one interviews with the participants and a series of teaching experiment sessions with two groups of three participants. Pre-interviews were conducted to determine students' current understandings of the various topics asked in the interview tasks (see Appendix A). More specifically, the first interview task, "Channels," involved the figures of various channel systems, some of which represent equiprobable routes, and the students were asked to identify which ones had equiprobable routes to exit 1 and exit 2, and to give an explanation of their responses. The second task, "Ice-cream," required students to list all nine possible combinations of three flavors of ice cream served in three different types of container. Next, in the "Swim Team" task, the students were asked to determine the probability of a single event as a ratio of the number of favorable cases to the number of all possible outcomes. Then, the "Stickers" task involved finding the most likely outcome based on the frequencies given in the table. In the "Marbles" task, the students needed to complete the list of the sample space by including all the permutations of two elements. The "Gumballs" task required students to determine a sample based on the proportions in the population. The final task, "Spinner," involved compound event probability that could be determined based on the sample space. During the interview sessions each participant was asked to show his or her work on the interview tasks sheets or explain it out loud, and thus some student artifacts were collected as supplementary document for the analysis of transcribed data from the interviews.

The student responses from the pre-interviews were analyzed quantitatively in terms of correct and complete answers and in terms of reasoning (the results are discussed in Chapter 5), to form two groups of students which were similar with respect to their prior knowledge. Starting from these current understandings and informal knowledge that students had, the teaching experiment sessions were conducted to address the supporting research questions stated in Chapter 2.

The teaching experiment consisted of eleven sessions lasting from 1 to 1.5 hours, over a period of seven weeks. I acted as a teacher-researcher in these sessions and the participants were always encouraged to share their ideas, conjectures, and methods, and to justify them. The instructional mode was a combination of individual and group work followed by a group discussion in a sequence of tasks that were designed to support students' emergent understandings and engagements in the context of reasoning about distributions in probability situations (see Appendix C).

For the teaching experiment study, thirteen tasks were designed and sequenced to address students' informal understanding of distribution and their reasoning about distributions, in which they develop such probability concepts as sample space, probability, combinations and permutations, probability distribution, equiprobability and symmetry, relative frequencies, and the law of large numbers. Chapter 4 discusses how these tasks were developed, sequenced, and revised in light of the pilot study. At the end of the teaching sessions, a post-interview was conducted with participants individually to determine their current understanding of topics and thinking processes in the tasks.

## Methods of Data Collection and Analysis of Data

The data consisted of video and audio tapes of the interviews and teaching sessions, the students' written work produced during the interviews and teaching sessions, and the field notes after the sessions. The digital camcorder was located at a place where it could capture the group interactions. When needed, the camera was operated by the researcher to focus on an individual student, a particular work at the board, or the computer screen during the simulations. To record my thoughts and reflections about each session, a notebook was kept. Ongoing analysis occurred between the sessions and a retrospective analysis focused on the cumulative episodes.

The analysis of data involved a qualitative analysis of pre- and post-interview data and ongoing and retrospective analysis of the teaching sessions, to characterize participants' reasoning about distributions in probability situations. In the analysis of the video recordings of the interviews, in order to formulate the models of students' probabilistic knowledge and reasoning, the focus was on (1) the way the students arrived at their responses (i.e., using a list, drawing, quantifying the probability of an event) and (2) their ways of reasoning (i.e., deterministic, causal, proportional, random, chance-based reasoning). Between the teaching sessions, ongoing analysis of the teaching episodes was conducted by watching the video tape of that day's episode and looking at the students' current probabilistic understandings and reasoning as well as the emergent ideas from the group discussions. When needed, the plan for the subsequent teaching session was refined (i.e., including some follow-up questions to clarify students' responses, questions eliciting different methods, and identifying possible probabilistic ideas and reasoning to be further examined). Finally, in the retrospective analysis of the teaching episodes based on the video tapes of all the sessions, the focus was on the actions, activities, strategies, inscriptions, and language the students used through the tasks.

To analyze the qualitative data upon the completion of the study, the data were organized by tasks and then the content of data from the interviews, teaching sessions, and student artifacts was examined for emergent patterns and themes as suggested by Confrey and Lachance (2000). The "grounded theory" (Strauss & Corbin, 1998) approach was used for this purpose. The analyses included doing a line-by-line analysis of each transcription along with the artifacts and videotape records. Sometimes a whole sentence or paragraph and sometimes a word or a phrase was analyzed. Short and quite simple memos were used to record initial impressions and thoughts for identifying the themes and categories in each transcript. After the initial coding, all transcripts and artifacts were reviewed to extract categories.

After identifying emerging categories, I sought patterns, along with their variations, in order to specify properties and dimensions of each category. In doing so, the constant comparison method (Strauss & Corbin, 1998) was used. The data available for each category in each transcript along with video data were repeatedly reviewed by comparing incidents applicable to each category. Then, the list of properties and dimensions were generated, if available, for each participant and sometimes for each group based on the responses, actions, and strategies they used. Next, axial coding was employed to build connections within categories and selective coding was used to identify the structural relationships between the categories, as suggested by Strauss and Corbin (1998).

Although the methodology of this study is informed mostly by a constructivist perspective, I believe socio-cultural contributions from Vygotsky's work are equally important in studying children's thinking. In this study, I paid attention to the relationship between the actions and activities of individual participants and those that emerged through the social interactions among the students, between the students and the teacher-researcher, and between the students and the mediational tools. Critical moments sometimes were identified ahead of time, when I planned to introduce a new language (i.e., the term "distribution") to them, or to provide a hint which could assist students' performance. Other times, I identified critical moments on the basis of the data, when I watched the video tapes of interactions, including myself (i.e., why I asked that question at that point). Hence, the three themes in Vygotsky's work discussed above provided me with an approach to the analysis of data collected in this study. In particular, taking a Vygotskian perspective on the notion of mediational means, I looked at what factors mediated the students' development of ideas about probability distributions and related concepts. What are the characteristics of students' behaviors in both acting upon something and in mediated activity? Moreover, the concepts of internalization and the ZPD were utilized in the analysis of participants' actions in their group work and the group discussions when the teacher-researcher or another student in the group, as a more knowledgeable person, guided a student on the procedures that the student could not perform without the assistance.

## Summary

Here I revisit the research questions of the study to link them to the theoretical bases, the design, and the method of data analysis which were discussed in this chapter. The four supporting research questions that were developed to unpack the main research question were as follows:

1) What are the students' prior knowledge about probabilistic concepts and probabilistic reasoning?
2) What kinds of informal knowledge and strategies can serve as starting points?
3) What are the conceptual trajectories that students take during the teaching experiment?
4) What are the resources (learned ideas) students bring into understanding of probabilistic concepts and reasoning?

The design of the study involved (1) one-on-one interviews to document each participant's knowledge and reasoning about probability concepts before and after the study, using Piaget's clinical method in order to address the first and fourth questions, and (2) the constructivist teaching experiment with a small group of students to address the second and third questions. Therefore, the methodology of the study mainly was informed by the constructivist approach, in examining the children's intellectual development over the course of the teaching experiment study. Moreover, the insights from the socio-cultural approach based on Vygotsky's work provided tools for examining the social interactions and the mediational actions during the teaching sessions conducted within small group settings. The implications of the theoretical bases discussed in this chapter are further delineated when the development of the conjectures is described in

Chapter 4. Last, in answering the research questions, the qualitative analysis of the data from the interviews and the teaching episodes probed into students' current knowledge and reasoning about probability concepts, their informal descriptions and reasoning about distributions, and development of probabilistic concepts.

Next, Chapter 4 reports on the findings of the pilot study which led to the revisions of the tasks and the development of initial conjectures for the study. Then, Chapter 5 presents the results from the pre-interviews. In Chapter 6, the retrospective analysis of the teaching episodes by tasks is discussed. Chapter 7 then documents the learned ideas over the course of the teaching experiment based on the analysis of the post-interviews. In the last chapter, the research questions are revisited again to be answered based on the findings presented in Chapters 5, 6, and 7. Chapter 8 also describes the conceptual corridor for students' reasoning about distributions in chance events and their development of probability concepts.

# CHAPTER 4

# THE PILOT STUDY AND
# THE CONJECTURE OF THE DESIGN STUDY

## Pilot Study

I conducted a pilot study prior to the teaching experiment study for two reasons: (1) to revise the tasks to be used in the teaching episodes; and (2) to develop conjectures about the conceptual trajectories along which students' ideas about distributions in probability situations develop. The pilot study spanned seven days and included one-on-one interviews with students and three hours of teaching sessions with the whole group. The participants were four fourth-grade students (9-year-olds), named Brad, Jim, Kate, and Tana (pseudonyms).

In the following subsections, I discuss the findings from the pilot study. These results also have been recently published elsewhere (Kazak & Confrey, 2006). Subsequently, I describe the insights gained from those findings which helped me further elaborate my conjecture and refine the tasks.

### Task 1: Distributions in Different Settings

To examine how students look at various distributions of things in different settings, I began with conducting one-on-one interviews with four participants. When I asked students to describe what they noticed in the pictures and whether they could see a pattern (see Appendix C for the tasks), their responses revealed some statistical aspects as well as causal explanations about the distributions (Table 1).

Table 1. Characteristics of students' explanations of natural distributions during the interviews in the pilot study.

| **Description** | **Sample Responses** |
|---|---|
| Variability (Spread) | "Most sheep gathered up together." (*Kate*)<br>"Over here they [buffalos] are kind of spacing out, but over here they look like jamming up a little bit." (*Jim*)<br>"They [flowers] are scattered." (*Tana*) |
| Typicality | "Most leaves are under the tree." (*Tana*) |
| Density | "They [bees] are in big crowds." (*Kate*)<br>"They [sheep] are less and there is like different spots where they eat. More here because they are all together." (*Brad*) |
| Causality | "There are just little [leaves] that made this far. Probably because the wind would have to be blowing long enough in the right direction for those get there. But that would happen to fewer leaves because mostly they would fall down by the trunk." (*Jim*) |

In most of the cases, students seemed to view distributions in given natural settings as clumps or groups when they talked about how the things were distributed in the pictures, where most of the things were located, and which regions were high/low density areas. Moreover, when asked to explain why there were such patterns, students often provided causal interpretations rather than explanations indicating a chance factor.

Following the interviews with individual students, I selected some of the pictures (a buffalo herd, a sheep herd, wild flowers in a plateau, leaves under a tree in the fall) to discuss with the whole group in the first session of the pilot study. In analyzing this episode, I mainly focused on students' reasoning about the density and the estimation of the number of things in the pictures. For instance, those who talked about the density used *bunch, pile,* or *crowd* to refer to high density areas and *separate, left alone,* or *spaced out* to refer to low density regions:

> *Brad*: There are more in the [right] corner.
> *Sibel*: How do you know there are more?
> *Brad*: Because there is like a *big bunch* together.
> *Sibel*: How about the others [in the middle]?

*Brad*: They are just *separate*.

When students began to talk about the picture of wild flowers in a plateau, I asked them whether there were more red flowers or blue flowers in the picture. Jim said that it was hard to tell how many red and blue flowers there were in the picture because "you can't really count these flowers." Then, I was curious about what students would think about counting the buffalos in the earlier picture:

*Sibel*: So how about if we go back to this picture [buffalo herd], can you count? How is it similar or different than this one [wild flowers]?
*Brad*: There is a lot of them.
*Kate*: They are bunched up.
*Sibel*: Can you count them?
*Kate*: Yeah, you can estimate. I'd say 270.
*Brad*: I'd put them in a cage.
*Jim*: If I could have something to mark, I would separate this into four pictures to cover every buffalo. And then count one group, write that down. Count the next group and write that down.

In this exchange, I found that each student had different ideas about estimating the number of buffalos in the picture. For example, Brad explained that he would put the buffalos in a cage in groups of hundreds for counting. However, Jim divided the picture of buffalos into (approximately) equal quarters to show his method of estimation.

*Sibel*: Do you think there are equal numbers of buffalos in each quarter?
*Jim*: I have doubt.
*Sibel*: Why?
*Jim*: Because over at this corner [left, upper] there is not much. Then on this side [right, upper] over here it look like there is more. This area [right, lower] doesn't hold too much. And on this side [left, lower] there is fair amount.

Jim's use of equal partitions to estimate the number of buffalos gives a sense of relative density. When he talked about the number of buffalos in each quarter, he contrasted the areas that held "more" or "fair amount" of buffalos with those that contained "not much." It seemed that Jim's method of clustering buffalos in about equal-sized sections entailed the concept of density which was not part of Brad's way to estimate the number of buffalos.

**Conjectures/Revisions**: In the pilot study, I used a variety of pictures, including distributions of animals in a field, fish under the water, flowers in a field, leaves under a tree, cookies with chocolate chips, and airline routes. Among those fourteen pictures, I selected four to be used in the teaching experiment. In doing so, I considered student responses to decide which of the pictured distributions best addressed the underlying concepts of distributions that were of interest.

One can talk about different kinds of distributions, such as distribution of certain measures (e.g., height, speed, and temperature), which would be distributed in two-dimensional space, and distribution of things in the nature, which would be distributed in 3-dimensional space. I selected pictures of natural distributions and representations that students may encounter in daily life, in order to discuss different aspects of distributions, such as spread, aggregates, clumps, likelihood, expectation, necessity, causality, and density. Initially, I conjectured that students might have informal ideas and language to talk about the distributions of things in different settings, such as animals in a field, leaves under a tree in the fall, and routes for an airline on a map. The existing literature documented that students expressed the qualitative characteristics of distributions as clumps, clusters, bumps, hills, gaps, holes, spreadoutness, and bunched-up-ness (Cobb, 1999) on graphical representations (i.e., stacked dot plots). The pilot study data revealed similar findings. Those included informal notions of distributions (i.e., "gathered up together", "spaced out", "jamming up", "in a pack", " groups", "scattered", "bunched up") as well as causal explanations (i.e., "more lines on this side, it looks like there are more places on this side", "more likely to fly from Las Vegas to Salt Lake city because it is closer", "the wind blows the leaves", "they [sheep in a group] are probably friends", "there is better grass there") to explain the patterns. Moreover, I found that students were likely to talk about density relative to the amount of space, especially as an attempt to estimate the counts. When asked what they noticed in the picture, students could note the patterns of arrangement of things in addition to the other things in the picture, such as what is in the surrounding area. However, I conjectured that they were more likely to care about patterns when they paid attention to the arrangement (the way they are distributed) of and quantity ("more" or "less") of things. With the modifications to the tasks, I expected to find out more about students' natural language for distributions, when they might care about the patterns, and what might lead them to talk about likelihoods in the actual study.

**Task 2: Dropping Chips Experiment**

The following sequence of activities is intended to support students' understanding of the notion of distribution in designed settings. In these tasks, students conducted various experiments in which they were asked to predict, generate, and interpret distributions of objects (see Appendix C).

For this task, I presented each pair of students with a number of blue, red, and white chips; a tube; a measuring tape; color markers; and a plain poster sheet. Then, students (in pairs) dropped 20 chips through a tube when the tube was 15 inches above the ground. While describing the distribution of the chips, students first focused on where most of the chips were on the floor by showing a hypothetical border around that middle region. For example, Jim said, “Because I was holding this [the tube] right about here. So, they kind of *stacked up* right here [showing a smaller area in the middle] but they are about around here [showing a larger area around that middle one].” Note that his explanation included the notion of density (“stacked up”) indicated by the small area right under the tube (i.e., the middle clump) and an expected variability shown by the larger area around that.

When students were asked to conduct the same experiment by holding the tube 30 inches above the ground, their predicted plot of the distribution of the chips would still be that middle clump which was a bit bigger, showing the density under the tube and more area outside of the middle one to accommodate the expected spread of the chips. After the experiment, Tana looked at the distribution of the (white) chips on the sheet and pointed to the region where a few chips landed all together under the tube: “this is like a pile and the rest is separate.” In Tana’s explanation, the “pile” indicated the middle chunk with higher density as opposed to the low density and more spread in the area outside of that middle chunk. Since the previous experiment results for the (blue and red) chips, which were dropped at 15-inches above, were still available on the floor next to the last white chips, I asked students to contrast each of the chip patterns. Kate responded that the white chips were more separate and scattered whereas the blue chips were gathered around in the middle. I asked them why the red and blue chips were more together but the white chips were separate. Kate hypothesized that the height at which the chips were dropped might affect the outcomes because the blue and red chips were dropped at 15 inches above the ground while the white chips were dropped from a higher position (“15 more inches”). Next, in the discussion about the effect of the height variable, students were able to offer reasons and conjectures for why the chips distributed that way. For example,

> *Jim*: When they landed, I noticed that a lot of them started rolling around. That might have affected it.
> *Brad*: If it goes up higher, then they will spread like *almost everywhere*.
> *Sibel*: Almost everywhere. Hmmm. That's interesting.
> *Tana*: Maybe a little bit around here [region under the tube] and the rest of them are *all over* [beyond the region below the tube].
> *Sibel*: So, do you expect more "all over the place" when you do it higher?
> *Kate*: Yeah. If it was like a foot-long, it would probably be close to each other.

The student responses, such as "almost everywhere" and "all over," could be interpreted with two different notions: a similar but larger distribution or a random spread of chips. Since I did not ask Brad and Tana to explain what they meant by those expressions, it was hard to speculate about their reasoning with "almost everywhere" and "all over." However, Kate's response indicated that she was able to make a conjecture about the closeness of the chips (density) at a given height at which they were dropped.

In the following activity, considering the effect of height for the distribution of the chips, each pair of students created their own game in which they gave different points to landing a chip near or far from a target. Each pair chose a higher position than 15 inches to drop the chips in their games, considering the effect of height on the spread. Jim for example said "we did it at higher level so they'd roll around more" but he was not sure if that would give a "higher or lower chance" to win the game at the moment. In Kate and Tana's game, they divided the sheet into four regions of different sizes as seen in Figure 4a. They assigned the highest point to the blue region at the lower right corner as Tana explained, "sometimes the smallest part is hard to get on." Kate added that once they did a spinner experiment with a yellow region comprising 1/8 of the whole and only three times did the spinner land on the yellow because it was the smallest area. It seemed that

Tana and Kate’s previous experience with a spinner task led them to generate a similar model which would work with the dropping the chips activity. In Jim and Brad’s game (see Figure 4b), the bigger circle in the middle had the lowest score while a very small area close to the point where the chips were dropped got the highest point in the game, which they called “the bonus point.” Jim and Brad also had an idea of “losing points” that were assigned to the two regions outside of the bigger circle for the chips expected to roll around randomly. The regions drawn in Jim and Brad’s game sheet looked very similar to their predicted inscriptions showing the distribution of the chips in the earlier activity. By including “bonus point” and “losing points,” Jim and Brad began to express different chances that might exist in a typical middle region and the outside of that region.

(a)

(b)

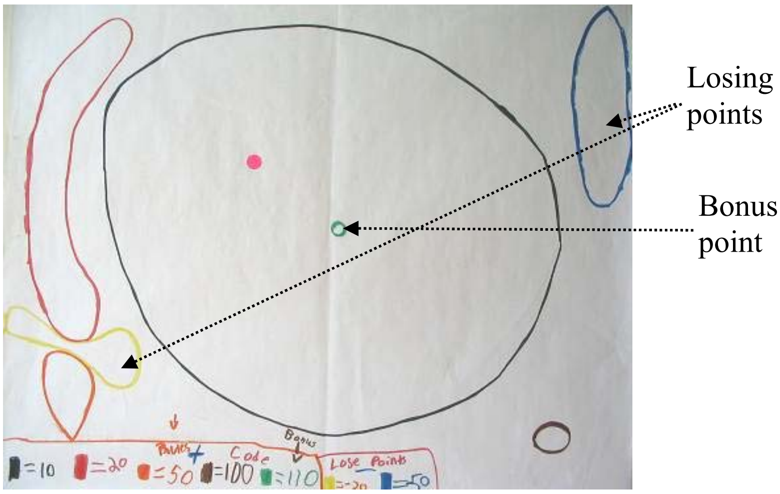



Figure 4. Student-generated games: (a) Kate and Tana's game (b) Jim and Brad's game.

**Conjectures/Revisions**: Through the chip dropping experiments, I anticipated that the notion of middle clump with varying density may come up in students' inscriptions of distributions (i.e. "more of them in this spot", "crowd" shown by a bigger circle, "stacked up right here" indicated by a smaller circle in the middle). In the pilot study, student strategies involved showing and drawing hypothetical borders around where most of the chips were expected to land. Therefore, in the actual study I asked participants to use the same sheet to show their predictions about the distribution of the chips for each experiment, using different color markers to contrast different results. When asked to drop the chips at a higher position, students tended to draw a bigger middle region and some small regions around that to accommodate the middle clump plus a bigger spread or more extreme data points (i.e. "this is like a pile and the rest is separate"). Drawing upon the student responses in the pilot study, I intended to investigate whether they thought the pattern would be similar but larger and whether they thought the chips would roll around randomly, and what "all over" meant to them if they mentioned it. In student-generated games with dropping chips I observed that they tended to assign a bigger score to the locations where the chips are not very likely to land, such as the blue area which was the smallest and not in the middle in Figure 3a and the green region which was close enough to the middle (where the tube was supposed to be held) but very small. Therefore, I conjectured that they might have some intuitive ideas about chance in games, such as a dart game, and so I included a discussion in the teaching experiment study about a *dart board*, scores assigned to different regions on the board, and more/less likely regions when throwing darts, prior to the designing the game task. On the basis of both the pilot data and the existing literature (e.g., Lehrer et al., 1994; Metz, 1998), I thought that students were likely to draw upon their personal experiences and to use a deterministic mindset, rather than to believe in pure chance in these activities. In order to investigate students' conceptions of chance in different contexts, I also included the *Gumballs activity* in the teaching experiment. In this task, students made predictions about the color of the gumball they might get from the mixture in the gumball machine. Whether students would approach the two activities (with the chips and the gumballs) differently was investigated in the study.

**Task 3: The Split-Box**

To investigate students' understanding of distributions that are generated with the notion of chance inherent in the physical apparatus, students were asked to experiment with an inclined box, I called "split-box" (Figure 5), with a centered funnel-like opening on the upper part to drop the marbles and a partition dividing the lower side into two same-size slots (Piaget & Inhelder, 1975). In this task, I focused on students' predictions about the outcomes, their interpretations of the arrangements of marbles, and how they reflected on the preceding observations.

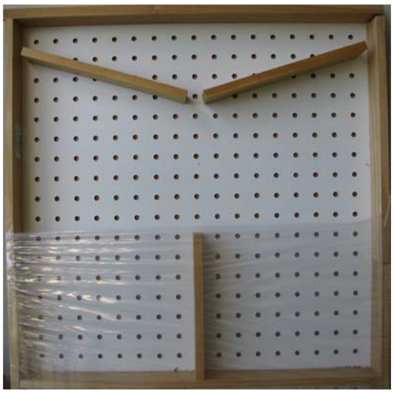


Figure 5. The split-box for marble drops.

Students began by making predictions about whether the marble would go to the left (L) or right (R) compartment when the marble was released right at the funnel-like opening.  Kate conjectured that the marble might go to the right without hitting the middle divider, but if it "bumps up" against the divider, then it might go to left.  In her reasoning, the outcome of the experiment depended on which path the marble would take when it was released from the top.  After the first result (L), students made predictions again.  Brad, who guessed R in the first experiment, made a prediction of R again (the reason to predict R again could be based on the previous experiment result with an expectation of R after L).  When the second result happened to be R, Brad made a conjecture about a pattern of results: L-R-L-R.  It seemed that Brad expected a pattern of alternating results which could be attributed to an understanding of randomness in chance events.

In the subsequent experiments, I asked students to drop 10, 50, and 100 marbles. Students carried out several investigations by letting those marbles fall down through the funnel and watching how they bounced off the middle divider to understand the mechanism of the physical apparatus and possibly to find out an algorithm to predict outcomes. To do so, they dropped different numbers of marbles from each side of the funnel, such as five marbles from each side, or six marbles on the left side and four on the right side, or all on one side. After several experiments, Jim hypothesized that “if we put more on this side [left], it has a bigger chance to go on this side [right] because they are opposites and it might go something like that and in this something like that [showing possible paths from the left-top to the right-bottom and vice versa].” The children (ages 7-11) in Piaget and Inhelder’s study (1975) also demonstrated an understanding of the mechanism in the apparatus when talking about possible trajectories the marbles might take as a result of collision of marbles and interactions in the mixture of marbles. When asked to predict the number of marbles in each side of the split-box, students tended to make their predictions unequal, but “close-to-even,” such as 6 to the left and 4 to the right, or 27 to the left and 23 to the right, or 49 to the left and 51 to the right. Although students used the notion of “50-50” to refer to the equal distribution of marbles in each slot when they dropped 100 of them, their predictions for the results were mostly “close-to-equal” (i.e. “48-52”) for 100 marbles. Similarly in Piaget and Inhelder (1975), children (7-11 years of age) expected about equal number of balls between the right and left slots, but with no recognition of any equalization as the number of balls increases, an issue which I did not examine in the pilot study task.

**Conjectures/Revisions**: The split-box demonstrates a single probability, L=R, and thus creates a uniform distribution of marbles. However, there is a much more complex situation than that. For example, the marbles may or may not hit the middle divider because of the way they roll down, depending on the box surface (smooth and flat, or not), the marbles (uniform and perfectly round, or not), and the force applied to the marbles (even, or not). Then, as stated by Piaget and Inhelder (1975), there is physics involved in this task. The pilot study data confirmed that. Students developed conjectures about how the marbles would roll based on the ways of releasing them from the top (i.e. dropping five marbles from each side or one on the left side and nine on the right side, or dropping them in a lined-up position or pouring all the marbles into the funnel). These investigations led them to make mechanistic arguments, such as a particular bias in the mechanism.

Like in the pilot study, the task started with an individual marble drop in the teaching experiment because I thought that students could watch the path a marble takes and think about how the set-up works. Also, I wanted to see whether their predictions would indicate a pattern, such as alternating outcomes based on the previous result or sticking to the previous prediction. Therefore, I modified the number of marbles dropped (ten marbles instead of four) in this activity since it would be easier to see the patterns in students' predictions with more trials (if there were any pattern).

When asked to predict the number of marbles in each side of the split-box for 10, 50, and 100 marbles, students tended to make their predictions "close-to-even." In another study, Vahey et al. (2000) found similar student responses to a different task with equiprobability. For instance, the majority of the seventh graders in their study responded that the coin was fair based on 47 Tails and 53 Heads in 100 coin-flips because it was close enough to even. Similarly, the students in this study believed that the result would not come out exactly 50-50 or half-and-half with the marbles in the split-box. In the revision of the task, I also included questions about comparing the sameness of the likelihoods of predictions, such as 6-4 vs. 4-6 with 10 marbles, as well as predictions for large number of marbles, such as 200, 500, and 1000, to elicit students' thinking about a possible bias to any side for a particular reason and any systematic prediction with the role of large numbers.

After the pilot study, I decided to create a game to be used in the actual study (see Appendix D). I called it "Multi-level Split-box Game" (Figure 6). A similar table board game was suggested to simulate the classical Galton Board using counters and coins in Ughi & Jassó (2005). In this game, students moved the counters representing marbles from the top to one of the compartments at the bottom by dropping a marble in the split-box for each step (the total of five). They were asked to mark the left and right turns on their counters after each marble dropping trial so that they could have a record of each path. I thought that the idea of expanding the split-box as in Figure 5 would provide students an elabroated model in which they could examine how binomial distributions were produced. Based on Ughi & Jassó (2005), I conjectured that students would be most likely to recognize (informally or qualitatively rather than quantitatively): (1) the symmetry around the center compartments at the bottom; (2) the presence of more counters (or marbles) in the center and few on the sides; and (3) the occurrence of the same number of left-turns in a given compartment.

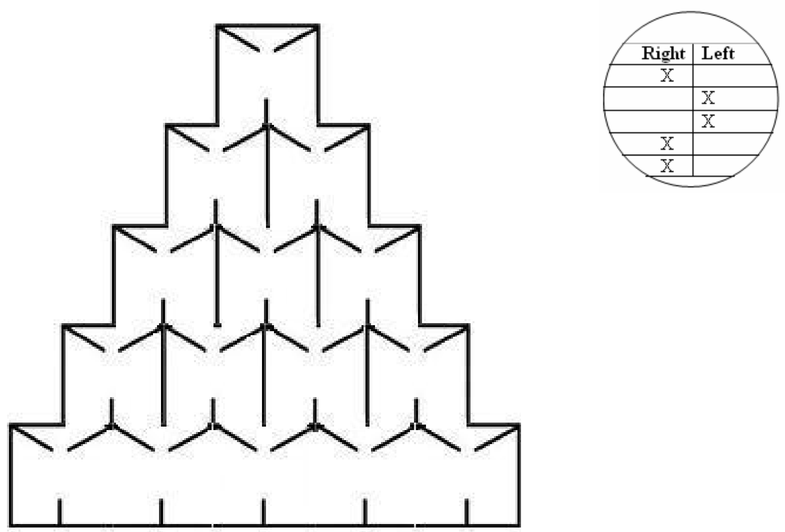



Figure 6. The Multi-level Split-box game board and the example of a counter.

**Task 4: Flipping a Coin**

Prior to the *Hopping Rabbits* task in which students were asked to simulate rabbit hops by flipping a coin, I wanted to examine their conceptions and reasoning about coin flips in repeated trials. Before the experiments with flipping a coin, students discussed the purposes of flipping a coin and the possible outcomes based on their personal experiences. They mentioned different purposes, such as to resolve a dispute and to make a decision particularly in sports, i.e., football, basketball, and baseball. With regard to the outcome, Jim said that it was “50-50 chance” like the marbles in the split-box except that the split-box was “not perfect,” referring to its mechanism.

Next, I asked students to make predictions about the outcomes before each coin flip. As seen in Table 2, there were different patterns in students’ predictions. For instance, Jim changed his prediction of Tails to Heads after 3 successive Tails whereas after the second Tails, most of the students said “Tails” except Tana who predicted Heads consistently. To be able to make any conjecture about their reasoning, I would have needed to ask them how they made their decisions, but I did not do this.

Table 2. Students' predictions for flipping a coin five times and the actual outcomes in the pilot study.

| **Students' Predictions** | | | | **Actual** |
|---|---|---|---|---|
| *Jim* | *Kate* | *Brad* | *Tana* | **Outcomes** |
| T | T | H | H | **T** |
| T | H | T | H | **T** |
| T | T | T | H | **T** |
| H | T | T | H | **T** |

After this brief experiment with flipping the coin, students were asked to just predict the outcomes of 5 coin-tosses and 10 coin-tosses. Similar to the findings in the previous task (the *Split-box*), students mostly tended to predict "close-to-even" results (i.e. 6 Heads and 4 Tails) while some believed in "extremes" (i.e., 1 Heads and 9 Tails or 10 Heads and no Tails) thinking that "anything could happen" based on the outcome approach (Konold et al., 1993).

**Conjectures/Revisions**: Coin flipping is another model for examining equiprobable outcomes. When making predictions before each coin flip, students' responses suggested that there might be some pattern-based reasoning in relation to their conception of randomness. In repeated trials, students tended to predict "close to equal" or "even" outcomes. In the teaching experiment study, I planned to ask participants how they made their predictions and also to pay attention to when they would believe "extreme" outcomes. The idea of the law of large numbers could be followed up here again to help students consider the role of a small number of trials in extreme outcomes. In the actual study, I also included some more prediction questions for a large number of trials, such as 50, 100, 200, and 1000.

In order to elicit students' use of "50-50" chance in the outcomes of coin flipping, a *Spinner Task* was added to the sequence of tasks in the study. It involved predicting the outcomes of a spinner with three equal-sized parts (yellow-red-blue) if they were to spin it 5, 10, 20, 30, 100, 300, 1500, and 2000 times. Based on the student responses in the pilot study with the marbles in the split-box, I conjectured that students would predict mostly "even" and "close-to-even" results.

**Task 5: Hopping Rabbits**

The purpose of this task is to introduce students to a situation that can be modeled by a binomial probability distribution and to link the observed frequency of outcomes to the probability of outcomes through a simulation of an uncertain phenomenon. I adapted the task from Wilensky (1997), in which one of the subjects created such a model in an attempt to make sense of normal distributions. I introduced the *Hopping Rabbits* problem to the students as "Suppose there are a number of rabbits on a land and each rabbit can choose to hop only right or left. For each hop, rabbits are just as likely to hop right as left. We want to know where a rabbit is likely to be after 5 hops."

First, students were asked to predict and then simulate where a rabbit would be likely to end up after 5 hops, in repeated trials, by tossing a coin. Students' initial predictions revealed a deterministic approach:

> *Jim*: If I were a rabbit, I'd know where I'd land.
> *Sibel*: You would know?
> *Jim*: Yeah, because I get to do it….Or, I could just tell the rabbit what happens next.

However, introducing the idea of simulation of the rabbit hops with coin tosses (i.e., students assigned Tails to the right and Heads to the left) helped students consider the chance effect on decision-making. Based on the number of hops, students first noted the range of possible outcomes (from -5 to 5 on the number line given that they start at 0). Their responses to "where do you think they are most likely to be after 5 hops?" showed some variation:

> *Kate*: I think most of them on this side [right]
> …
> *Jim*: One. I think it is going to be this.
> …
> *Brad*: Three.
> …
> *Kate*: More here [on four].

Kate's last prediction "4" related to the most likely outcome after 5 hops led to a new discussion about whether it would be possible to land on an even number on the number line after 5 hops. Jim's strategy was to try different combinations of five hops to the left and right (see the paths in Figure 7) to convince others that it was impossible to land on even numbers on the number line after an odd number of hops.

After each group conducted their simulations and plotted their outcomes on the graph paper (see combined results in Figure 7a), they were asked to interpret them. Their responses involved comparing individual points, e.g.,"-1 has the most" or "1 is the second," as well as aggregates of data, e.g., "There is a majority in the negative side than the positive side." Moreover, Tana made a conjecture that since there were more rabbits on the negative side, the coin landed on Heads more than Tails (they assigned Heads to left initially). Students also noted that the outcomes were "spaced out" on the graph, which was due to the nature of a discrete random variable that students were asked to model in this task. They acknowledged the likelihoods of different outcomes referring to them as "easy to get," "hard (or rare) to get," and "equally easy/hard to get (or symmetric).” Jim even started talking about different ways to get to the places on the number line when he argued “usually to get to negative one, you want THTHH and it’s only 3 Heads and 2 Tails or sometimes it went HHTTH.” When they attempted to quantify the likelihoods of outcomes by figuring out the possible ways to get an outcome, students made use of different forms of “inscriptions” (Latour, 1990), such as lists, paths, and stacked plots (Figure 7a and b). Note that Jim’s list of combinations to obtain each outcome in Figure 7b is a critical step which constitutes operative quantification of probabilities when followed by recognizing the respective ordered arrangements of those combinations in a sequence. This example might add a new level of understanding about constructing an idea of chance and probability which, according to Piaget and Inhelder (1975), essentially depends on the ability to use combinatoric operations in random mixture cases.

(a) (b)

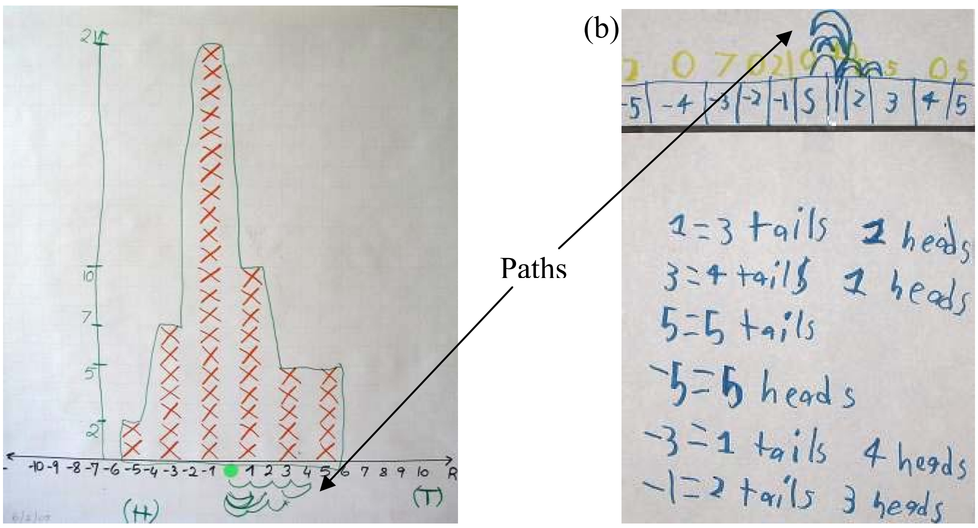



Figure 7. Student-generated inscriptions for the rabbit hops in the pilot study.

**Conjectures/Revisions**: In the *Hopping Rabbits* task, students’ initial responses indicated a distinction between the role of flipping a coin to simulate random rabbit hops and that of deterministic decision making. Based on the number of hops that are simulated by a coin toss, students noted possible and impossible outcomes. The outcomes of this experiment were described as “spaced out” since the random rabbit hops were discrete variables. Students tended to talk about the likelihood of outcomes in terms of “easy to get”, “hard (or rare) to get”, “equally easy/hard to get (or symmetric).” In finding all possible ways to get an outcome (i.e. combinations of 3H and 2T or 1H and 4T; ordered arrangements of those in a sequence; see Table 3) to quantify those likelihoods, students made use of different forms of inscriptions, such as lists, paths, and stacked plots. I believe that an understanding of notions of permutations and combinations are necessary for quantification of likelihoods. Hence, in the study, I emphasized both understandings of permutations and combinations and conceptions of relative frequencies, to link the likelihood of outcomes to the theoretical probabilities.

Table 3. The list of combinations and permutations of Heads and Tails for five hops and the final position after five hops.

| | **Combinations** | | | | | |
|---|---|---|---|---|---|---|
| | **3H2T** | **3T2H** | **4H1T** | **4T1H** | **5H** | **5T** |
| **Permutations (All possible ways)** | HHHTT | TTTHH | | | | |
| | THTHH | HTHTT | | | | |
| | HTHTH | THTHT | | | | |
| | HHTHT | TTHTH | | | | |
| | THHTH | HTTHT | | | | |
| | HTHHT | THTTH | HHHHT | TTTTH | | |
| | THHHT | HTTTH | HHHTH | TTTHT | | |
| | TTHHH | HHTTT | HHTHH | TTHTT | | |
| | HTTHH | THHTT | HTHHH | THTTT | | |
| | HHTTH | TTHHT | THHHH | HTTTT | HHHHH | TTTTT |
| | **1** | **-1** | **3** | **-3** | **5** | **-5** |
| | **Final Position = # of H - # of T** | | | | | |

Student responses in the pilot study revealed the importance of permutations and combinations in understanding the probabilities, particularly in transition from noting all possible ways to get each outcome to quantifying the likelihoods. Therefore, to examine students' initial thoughts on these topics in the teaching experiment study, I gave them the *Bears Task* before the *Hopping Rabbits* activity (see Appendix D). This task included hands-on materials (blue and red bears), paper, and markers. Students were asked to arrange five bears in a row as many different ways as they could when they were given five red bears and five blue bears to choose from. In this task, there are six different combinations, i.e., 5B, 4B1R, 3B2R, 2B3R, 1B4R, and 5R, and a total of thirty-two different permutations, like those of Heads and Tails in Table 3 above.

The pilot study data convinced me to include a computer simulation of the *Hopping Rabbits* in order for students to conduct large number of trials for the rabbit hops. This computer simulation was used (1) to run more trials (up to 10,000) for five hops and 10 hops; (2) to watch a particular rabbit while it is hopping; and (3) to change the chance of hopping to the right. For these purposes, I used the NetLogo Model "Binomial Rabbits" (Wilensky, 1998) which was modified[1] to include certain features that I wanted students to explore. The NetLogo is a programmable modeling environment for simulating natural and social phenomena, in which users can create their own models (Wilensky, 1999). Yet, it is simple enough that students can easily run simulations using the existing models. Students can open simulations and play with them exploring their behavior under different conditions. For instance, the NetLogo model used to simulate the rabbit hops in this study is shown in Figure 8. The interface of the NetLogo environment in Figure 7 includes the *view* (the black screen is where the action happens, like rabbits hop), the *sliders* (these represent the things that vary, such as the number of hops, number of rabbits, the chance of hoping right); the buttons (these start

[1] Izzet Pembeci, Ph.D. in Computer Science, provided the NetLogo coding to make the modifications to the model (Wilensky, 1998).

the particular actions like setting the view up [setup], starting the simulation [go], having the rabbits hop one at a time, starting and stopping to watch the hops of a particular rabbit); and the monitors (these display the values, such the numbers of rabbits on the left and on the right, the number each rabbit represents in the view screen, and the number of hops done). In Figure 8, for example, the NetLogo environment is set up for 10 rabbits that hop 5 times starting from 0 point.

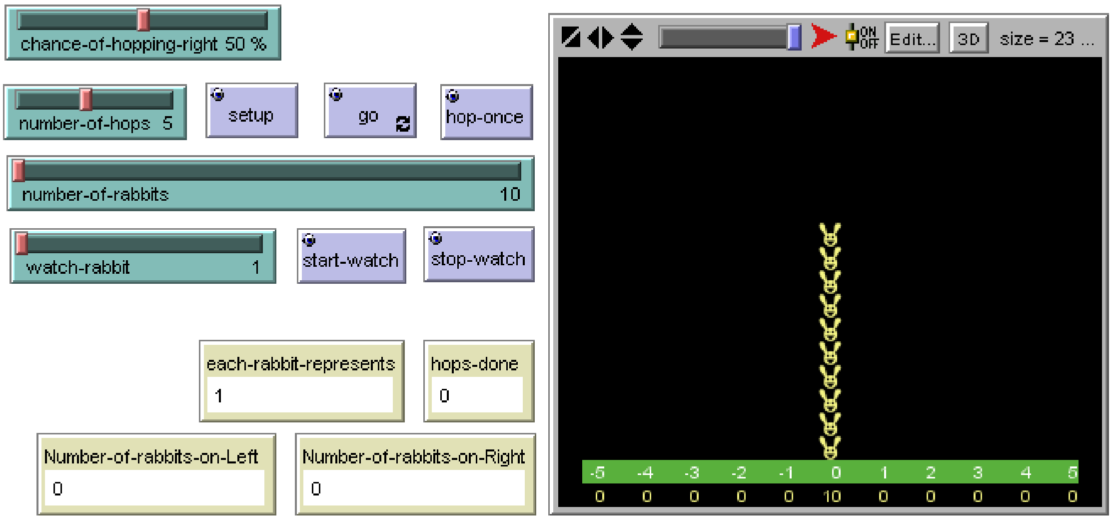



Figure 8. The NetLogo interface: Hopping Rabbits Task.

I conjectured that experience with a large number of trials in the NetLogo simulation would help students recognize certain characteristics of the distribution of rabbits, such as that there was symmetry around the middle (0), that most rabbits were close to the middle, that less rabbits were further from the middle, and that the fewest of them were at the opposite ends. Then, students could be prompted to discuss why there were more rabbits on 1s than on 3s and to compare the likelihoods of each outcome. They might use the paths (like the students in the pilot study did) to find out different ways to get each outcome, such as 1s, 3s, and 5s. Also, producing a list of all possible permutations of Heads and Tails for each combination (see Table 4) seemed like a promising alternative approach to find out all possible ways to get to each final position, since it was directly connected to students' experiences in the prior task (*Bears Task*).

Moreover, in the actual study, to elicit students' thinking about the likelihoods of outcomes in other familiar contexts, such as rolling dice, students were asked in a subtask to predict and explain the likelihood of certain outcomes in rolling a die and to predict the sum of the two dice, the distribution of which has a triangular shape and is a continuous variable. Dice were available for possible experiments. The goal was to ensure that students know that not all cases were "50-50" or equally likely. Hence, they were given an opportunity to experiment with two dice.

Finally, I conjectured that students' experiences with the *Split-box* task, the *Multi-level Split-box game*, and the *Hopping Rabbits* task could be followed up with the idea of the Galton box. The Galton box is a chance device that creates a particular probability distribution (a binomial distribution) and mostly is used to illustrate the process that gives rise to the shape of the normal distribution (the bell curve). This idea of illustration goes back to Sir Francis Galton (1899). When marbles are dropped from the top, they pass through a series of pins until they hit the bottom. The final position of each ball is determined by a number of independent, random events that make the ball go either to the right or to the left of the pin.

When designing this task, I used the NetLogo Galton Box model (Wilensky, 2002) with some slight changes[2] with the interface and the features of the simulation in the teaching experiment study. In the NetLogo environment, the *Galton Box* contains several rows of equally spaced pegs arranged as a triangular array (see Figure 9). Balls

[2] Izzet Pembeci, Ph.D. in Computer Science, provided the NetLogo coding to make the modifications to the model (Wilensky, 2002).

are dropped from the top, bounce off the pegs, and stack up at the bottom of the triangle. In this model, the chance of the ball bouncing right when it hits a peg is 50%. In addition to the features available also in the NetLogo simulation of *Hopping Rabbits*, such as the *sliders* and *buttons*, the Galton Box model has two other options: *shade-path* (it has the balls record their path as they fall down the triangle; for defining the most used path) and *pile-up* (it controls if the balls create piles or simply disappear when they reach the bottom of the triangle; for running a simulation with a large number of balls). The simulation task in this study (see Appendix D) included the Galton box model with (1) 1 row of pegs (similar to the *Split-box*), (2) 5 rows of pegs (similar to the five rabbit-hops), (3) 10 rows of pegs (analogous to 10 rabbit-hops), and (4) "chance of bouncing right"=50%, 75%, and 25%. In this task, students were asked to make predictions, run the simulation, and discuss the resulting distribution by comparing them with the predicted outcomes.

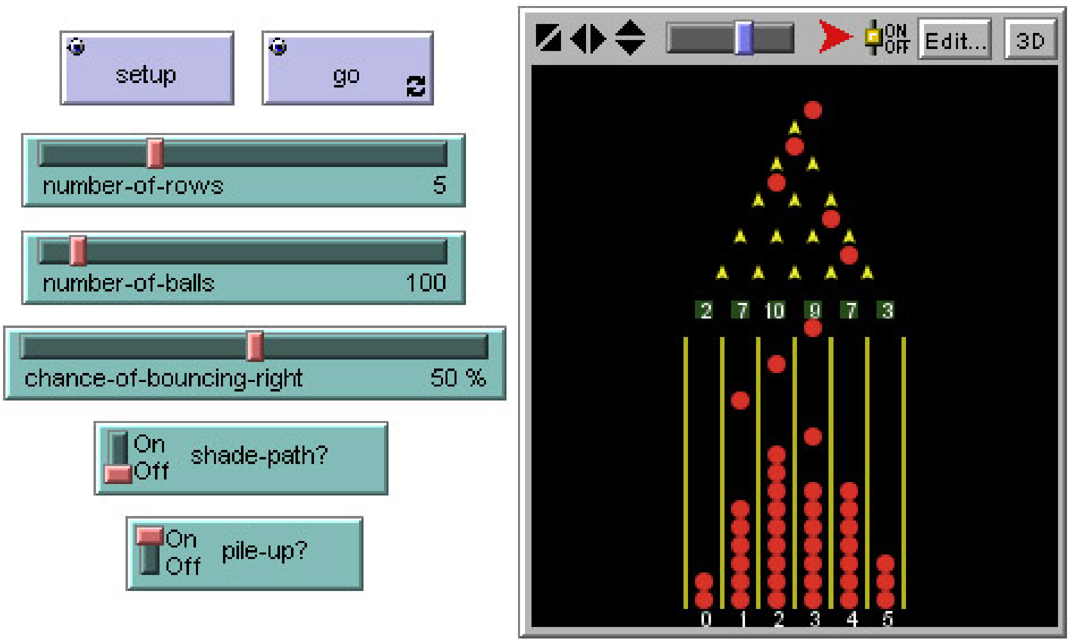



Figure 9. The NetLogo interface: The Galton Box Task.

## Development of Conjecture

As mentioned in Chapter 3, developing conjectures as starting points and testing them are important components of design studies. Confrey and Lachance (1999) described *conjecture* as "an inference based on inconclusive or incomplete evidence" rather than "an assertion waiting to be proved or disproved" (pp. 234, 235). The initial conjecture that guided my research involved the notion of distribution as a conceptual link between chance and data. I developed this conjecture on the basis of the relevant literature, the epistemology of probability theory, and the constructivist and socio-cultural philosophies as discussed in the previous chapters. Moreover, the analysis of the pilot study discussed above helped me further elaborate that conjecture. In the paragraphs below, I explain my conjecture in relation to two dimensions: *a mathematical content dimension* and *a pedagogical dimension* linked to the content aspect (Confrey & Lachance, 1999).

In this study, the *content aspect* of the conjecture included how the notion of distribution could be used to introduce the ideas of probability theory to students as early as in elementary school, rather than treating data analysis and probability theory as separate topics. As I laid out in Chapter 1, the two treatments of distribution in the literature, one viewing data distributed across all outcomes as aggregate in data analysis, and the other focusing on data distributed across all possible outcomes in association with their probabilities, led me to develop the content dimension of my initial conjecture. I believe that those two notions can be used to link discussions of probability theory and data analysis when students have an opportunity to engage with the fundamental ideas of probability theory, such as randomness, sample, sample space, probability, combinatorics, probability distribution, equiprobability and symmetry, relative frequencies, simulation, and the law of large numbers, as they engage with reasoning about distributions. Similarly, how an understanding of distribution connects and affects understanding of statistical concepts, such as the mean, median, spread, and shape, and how it relates to other kinds of statistical reasoning, such as reasoning about variation and covariation, have been documented in relatively recent studies in statistics education (see Cobb, 1999; Lehrer & Schauble, 2000; Cobb, McClain, & Gravemeijer, 2003; Shaughnessy, Ciancetta, Best, & Canada, 2004; Petrosino, Lehrer, & Schauble, 2003). With regard to probabilistic reasoning, some researchers (see Lehrer, Horvath, & Schauble, 1994; Vahey, 1997) examined students' ideas about randomness, sample space, relative frequencies, probability distribution, and the law of large numbers, across various contexts including model-based reasoning (i.e., using models of probability to reason about uncertain events) and fairness (i.e., equal chance of winning in a game). In doing so, the researchers used a *distributional approach* to probability (Moore, 1990) in a sense that probabilities result from the patterns emerging from many repeated trials of an event.

Furthermore, in the epistemological characterization of probability, the relation between the concepts of probability theory (in its theoretical form as a mathematical model) and the data (in its empirical form of relative frequency) is a dual interplay (Steinbring, 1991). In other words, probability (i.e., mathematical model) and data (i.e., empirical applications) can only be understood in relation to each other. For instance, the concept of equiprobability can simultaneously be interpreted as "the ideal mathematical description of equally distributed in the arithmetical sense, and the empirical representation of statistically equally distributed, as it emerges in concrete experiments"

(Steinbring, 1991, p. 148). To illustrate Steinbring's statement about the dual meaning of probability, let's consider the fairness of a six-sided die. From the mathematical modeling perspective, a fair die has an ideal, arithmetically equal distribution of outcomes, which refers to the probability of getting each outcome (from 1 to 6) as 1/6. From an empirical and statistical perspective, one can judge the fairness of the die on the basis of the observed outcomes in many repeated trials of concrete experiments by looking at the degree of deviations in the observed frequencies from a uniform distribution. Drawing upon this epistemological analysis of probability concepts, in this study I focused on the notion of distribution in chance experiments as a way to discuss the empirical situation in relation to its mathematical modeling (e.g., see *Hopping Rabbits* task).

The *pedagogical dimension* comprised the design and sequence of tasks and resources to be used in the teaching episodes. When designing and ordering these tasks and resources, I relied on the dual nature of probability concept, previous related research studies with children, and constructivist and socio-cultural perspectives about children's intellectual development (see Table 4). Since the focus of the teaching experiment was the notion of distribution in probability situations, I wanted to start with an informal discussion of distributions which students could experience in daily life. Also from the constructivist perspective, the tasks were designed to be constructed on students' previous knowledge and experiences. In these tasks, my goal was to reveal students' qualitative and intuitive ideas that could be developed into formal conceptions. At the end of the *Dropping Chips Activity* I started using the term "distribution" when I asked students to talk about what was common across all these distributions of chips. By this time students had informal language to talk about distributions in various ways. Since these activities foster an understanding of distributions with clumps and spread, I believe there needs to be a shift in language from informal to a shared language with which to talk about these distributions at this point. Thus, I discussed with students the idea of developing a common language, such as distributions, and I continued to use that term consistently.

Table 4. Synopsis of the sequence of tasks used in the teaching experiment study.

| *Tasks* | *Conceptual Themes* | *Context* |
| --- | --- | --- |
| **1. Distributions in Different Settings**<br>• Patterns of things in the pictures<br>• Arrangement of things in the pictures<br>• Reasoning about the ways the things areorganized or distributed and why | Distribution, spread, typicality, aggregates, density, likelihood, necessity, causality | Pictures of distributions in nature<br>Distribution of locations in an airline routes map |
| **2. Dropping Chips Experiment**<br>• Predictions concerning distributions of chips dropped through a tube<br>• Generation of distribution of chips on the floor<br>• Effect of height on the distribution of chips<br>• Anticipated effect of # of chips on the distribution of chips<br>• Comparison of the expected and actual distributions | Centered distribution, spread, middle clump, density, causality, randomness, likelihood | Physical situation |
| **3. Dart Game**<br>• Analysis concerning scores assigned around the target point<br>• Discussion about what to do to get the highest score | A game of skill vs. chance | Game |
| **4. Design Your Own Game**<br>• Designing a game with dropping chips<br>• Noting effect of height and # of chips on the distribution of chips<br>• Quantifying the chances for landing near or far from the target | Distribution, likelihood | Physical situation<br>Game |
| **5. Gumballs Activity**<br>• Predicting the color of a gumball | Chance, sampling | Random mixture |
| **6. The Split-box**<br>• Predictions concerning the distribution of marbles in the box<br>• Experiments with different numbers of marbles in repeated trials<br>• Projections concerning the previous results with the marbles for the following predictions | Uniform distribution, chance, physics, law of large numbers | Physical apparatus |

Table 4. (Continued) Synopsis of the sequence of tasks used in the teaching experiment study.

| *Tasks* | *Conceptual Themes* | *Context* |
|---|---|---|
| **7. The Multi-level Split-box Game**<br>• Anticipated results of playing with the marble and the split-box for moving counters on the multi-level split-box board game | Centered distribution, symmetry, combinations and permutations | Physical apparatus<br>Board game |
| **8. Bears Task**<br>• Arrangement of five bears selected from blue and red bears in a row | Arrangements, combinations and permutations | Combinatorics |
| **9. Coin Flipping Activity**<br>• Predictions about the outcomes of a coin-flip in repeated trials<br>• Anticipated results of flipping a coin in large number of trials | Equiprobability, notion of 50-50, sample space, law of large numbers | Random-generating device |
| **10. Spinner Task**<br>• Predictions concerning the outcomes of the spinner (with three equal parts) activity for different numbers of trials | Equiprobability, law of large numbers | Random-generating device |
| **11. Hopping Rabbits Activity**<br>• Simulations of binomial rabbit hops<br>• Reasoning about a probability distribution<br>• Quantifying the likelihood of outcomes (final position after five hops)<br>• Figuring out combinations and permutations | Binomial distribution, symmetry, law of large numbers, simulation, sample space, combinations and permutations, skewed distributions | Random-generating device<br>Computer simulation |
| **12. Rolling a Die and Sum of Two Dice**<br>• Comparing and quantifying the likelihood of outcomes<br>• Figuring out combinations and permutations | Sample space, combinations and permutations, uniform distribution, triangular distribution | Random-generating device |
| **13. Galton Box**<br>• Simulations of a binomial distribution<br>• Reasoning about a probability distribution<br>• Quantifying the likelihood of outcomes | Binomial distribution, symmetry, law of large numbers, skewed distributions | Computer simulation |

As discussed in Chapter 2, the dual property of probability was an important development in the emergence of the concept of probability. This property implied two notions of probability: the epistemic notion, that is, probability based on our imperfect knowledge or personal judgment of an uncertain event; and the statistical notion, that is, probability based on stable frequencies of certain outcomes and symmetry of games of chance (Hacking, 1975; Hald, 2003). Therefore, it made sense to develop tasks that could give an opportunity to begin with probabilities based on personal knowledge and judgment which could be checked with the empirical situation, as suggested by Steinbring (1991). Also, by proving students with task experiences in a wide variety of contexts, such as a physical situation, a physical apparatus, a game, a random-generating device, and a computer simulation, I aimed to reveal their understandings of various applications of probability concept.

The tasks (see Appendix D) involved the participants' predictions and explanations of patterns in distributions of things in the pictures and of chips (Tasks 1-4) as well as in outcomes of a chance experiment across many repeated trials (Tasks 5-13). Especially in the latter tasks, I included probability situations in which a single outcome was uncertain, e.g., flipping a coin, but some regular (global) patterns emerged in long run, e.g., the outcomes of five coin-flips (5H, 4H1T, 3H2T and so on; see Table 4). This aspect of the tasks is especially important because the distributional probabilistic reasoning cannot exist without recognizing the patterns in uncertainty (Metz, 1998).

The nature of the tasks in general involved students' active engagement in predicting, generating data, and interpreting the resulting outcomes by comparing them with their predictions in given random situations. This approach has been previously used by other researchers as an effective procedure in fostering students' development of the ideas of chance and probability (e.g., Fischbein, 1975; Lehrer et al., 1994; Metz, 1998; Piaget & Inhelder, 1975; Shaugnessy, 1992; Vahey et al., 2000). Moreover, in this study participants were encouraged to model the probability experiment by using simulations. As recommended by Shaughnessy (1977), these simulations involve (1) modeling a random situation by using an apparatus with known probabilities, such as coins, dice, spinners; (2) conducting experiments with many trials; (3) gathering, organizing, and analyzing data; (4) quantifying empirical outcomes, such as frequencies; and (5) making inferences from the empirical results. Furthermore, along the lines of Biehler's (1989, 1991) suggestions about the integration of computer supported experiences in probability education, the use of computer simulations provides students opportunities to conduct large number of trials as well as to utilize certain given features (e.g., changing the probability of hopping right for the rabbits and bouncing right for the balls, watching an individual rabbit hopping, and shading the paths of the balls in the Galton box) in order to explore important concepts like probabilities and combinatorial analysis of the paths. In addition, as argued by Vygotsky, tools like computer simulations have a potential for shaping thought and communication as *mediational means* (Wertsch, 1991).

# CHAPTER 5

# ANALYSIS OF PRE-INTERVIEWS

The purpose of pre-interview was to evaluate each participant's initial knowledge about probability and probabilistic reasoning. Each interview session spanned 12-36 minutes on the week of October 17, 2005. All students used paper and pencil to organize their thoughts and show their solutions. The pre-interview consisted of the tasks listed in Appendix A. The content of these tasks involved various topics, such as identifying the equiprobability routes in a physical apparatus, listing sample space and all possible combinations, determining the probability of a single event and the most likely outcome based on given frequencies, determining a sample based on the proportions in the population, and understanding of a compound event. In the following subsections, I discuss my understandings of these students' conceptions on each task domain.

## Pre-Interview Task 1: Channels

In the first task, an adaptation of Fischbein et al.'s (1967) set of boards including systems of channels with slight changes, I presented to the participant the following hypothetical situation:

> There are five figures showing the different channels (see A, B, C, D, and E). Suppose we place a marble at the top and let it drop many times one after the other, and then it will come out at the bottom in one of these numbered exits. Circle the figures where you think that the marble is as likely to come out of exit 2 as exit 1. Explain how you got that answer.

In the channel system A (Figure 10), the probability of coming out of either exit is 1/2 when the marble is dropped from the top and arrives at the middle of the two channels. The channel system B is a more complex version of A and the probability of the marble coming out of any one of the exits is 1/8, thus still equal for exit 1 and exit 2. However, the channel system C represents a situation that the probability for the exit 1 is ½, but the probability for the exit 2 is ¼ since the path leading to exit 2 is the outcome of the multiplication of probabilities, such as P(Exit2)=1/2*1/2=1/4. In the channel system D, the lengths of the channels are different on the right and left sides, but the marble is equally likely to go down through any of the exits. Finally, the channel system E represents a model of the addition of probabilities that implies that the probability of the marble dropping through exit 2, i.e., P(Exit2)=1/4+1/4=1/2, is greater than the probability of the marble coming out of exit 1, i.e., P(Exit1)=1/2*1/2=1/4.

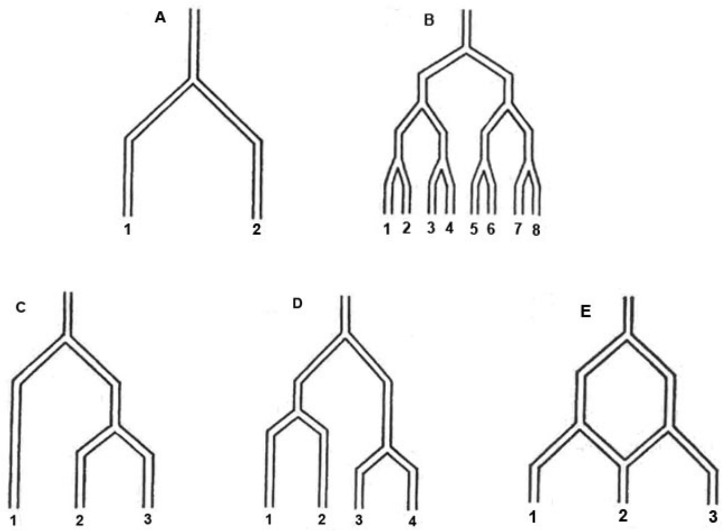



Figure 10. The figures (A, B, C, D, and E) shown in Pre-Interview Task 1: Channels.

A second interpretation of the channels in this task is both deterministic and mechanical. By this, I refer to a physical or mechanical explanation a student tries to offer based on, for example, his or her perception of minor visual differences on the channels represented in the figures.

Given the mathematical analysis of the situation presented to the students in this task, only Emily determined that the channel systems A, B, and D represent the equal probability of the marble coming out of exit 1 and exit 2. Her reasoning for A was in a probabilistic form when she said: “because if it is just in here, it could go this way or this way because they are both like, they are going different direction but they are both like the same way.” She made her decision for B for the same reasoning when she showed the likely paths leading to exit 1 and exit 2 on the paper with her finger. Similarly, she followed the possible paths leading to exit 1 and exit 2 on figure D and explained that she circled figure D “because it goes down and it can go either way. It comes down here and it can go either way, so it is probably just as likely.” However, Emily used the mechanical aspect of the channels, like some children in Fischbein et al.’s (1967) study did, with a deterministic interpretation when she responded that the marble would come out of 2 more often in figure C because, due to the bias slant towards right, “this comes out here (R) and hits that point then go that way (2).” Finally, she correctly identified unequal probabilities in figure E with the explanation indicating an intuition about the number of ways to get an outcome: “because it is like, it is probably easier for it to go down that way than that way because it is like there is two different ways it can come down that way (2). And there is only one for number 1.”

Alicia correctly identified the figures A and D in which there was an equal chance of getting the marble out of exit 1 and exit 2. In both cases, her reasoning was in a probabilistic form when she responded “it might go this way or that way.” However, she did not select figure B for the same reason because she thought that the marble could go any one of the exits (1 through 8) with no recognition of equiprobability. Perhaps, due to the greater complexity of the channel system, she interpreted that one could not know where the marble would go in this situation. Alicia noticed the case of unequal probabilities in figure C by pointing out that on the right side the marble could either go to exit 2 or exit 3, and thus exit 1 is more likely. Here she was using the probabilistic reasoning again when interpreting the situation as an uncertainty rather than a deterministic event. Then in figure E, she thought that exists 1 and 2 had equal chances without noticing the two different ways leading to exit 2. It seemed that she failed to notice an additive probability in this situation.

Caleb selected figures A and B as examples of equal probabilities for exit 1 and exit 2 by saying, "both the same because they are both equal." He thought that it would not matter which way the marble would come because the channels looked even (perceptually). I interpreted his response as mechanistic rather than probabilistic because in the video clip, he was actually examining mechanical aspect of the system for certain paths. For instance, his verbal response clearly indicated a mechanistic explanation for unequal chances for exit 1 and exit 2 in figure C: "[Looking closely at the figure] I see it. That doesn't exactly have a likely chance to come down here though. [*What do you mean?*] Like if you look real closely, this is slightly more up here than. It is not even. This is more up and this is more down…So if it goes there, it will hit this and it keeps going down there [R]." Even though the figure D represents equal probabilities for exit 1 and exit 2, Caleb did not select this one because of the similar mechanistic reasoning he used previously. In the case of E, it was more evident that he was focusing on the mechanic aspects of the channels when he said that the marble would come out of exits 2 and 3 more often than 1 "because it is tilted the same way as this thing's tilted [figure D]. Except this is more right here [right channel] and this is more down [left channel]. So it come down here but then if it rolls, it is going to like skip it, then it slows down it but it can't jump it so then falls down right here (2)."

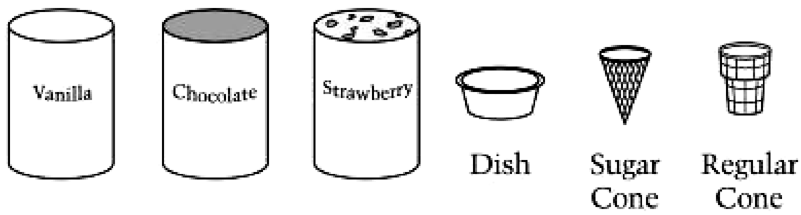
When selecting figure A, Josh said, "basically if you put a marble down, it can go either way." However, he further explained that by saying "It is not that force is on one side or another you know." Like Caleb, Josh's interpretation of the situation B involved the using of deterministic reasoning: "because it is already moving that way so it is most likely going to come out that way [1]" and "It would most likely to go to 1 because of the forces, the marbles have been rolling that way for a little while." Moreover, he chose exit 1 as more likely outcome in figure D because of a similar deterministic interpretation of the phenomenon. In figure C with unequal probabilities for 1 and 2, Josh argued that the marble could go either way (1 or 2) because he thought that event $A_1$ (marble coming out of exit 1) and event $A_2$ (marble coming out of exit 2) were independent of each other. For example, his explanation was: "It could go to either way because they both separate. Because first it is just going straight and then you don't know whether it is going to that way [right] or that way [left]…because they are off of different branches you know. That's most likely that could go either way." Next, his response to figure E was correct in that the marble would go to 2 more often. However, his reasoning was away from a probabilistic form, and more like a mechanic argument: "I think it would go to 2 because it would probably you know come down here [the middle of channels 1 and 2] and bounced off that wall a little bit, so it would be going that way [channel 2] and the same here [the middle of channels 2 and 3]."

Maya identified figure A with equiprobability by her response that either way was likely for the marble to go down. However, she failed to notice the equiprobability in figures B and D due to her deterministic reasoning. For instance, she chose exit 2 as more likely outcome (in both cases) with an explanation for D: "Because I think if it bumped here, it would go that way [left] and if it bumped here, then it would go this way [down] and if it bumped here, it would go that way [2]." For the situations with unequal probabilities, like in C and E, Maya's responses were inconsistent which I thought was due to the nature of her reasoning. In the situation of C, her perceptual reasoning, i.e., "because 1 doesn't have that many arms," led her give a correct answer [exit 1], whereas in figure E she thought that the marble would go more often to either 1 or 3 based on the deterministic reasoning: "I figured out that this way [either 1 or 3], probably going one way, probably or it might just go here a couple of times [2]."

Alex's mechanistic reasoning about the channels led him either determine the bias in equal probability situations, like in A, B, and D, or assess unequal probabilities in figures C and E incorrectly. Some examples from his responses in which he used non-probabilistic reasoning were the following: [figure D] "I think it would go in to number 1… because it is tilting towards [exit] number 2 so it has more room for the marble to go in to [exit] number 1" and [figure C] "it is more likely that the marble would go in to [exit] number 3 because [exit] number 3 has a wider space."

These findings show that Emily came into the teaching experiment with the ideas of equiprobability, multiplication and addition of probabilities (limited to the physical or visual representations) and probabilistic reasoning. Although Alicia's reasoning was probabilistic throughout the task, she failed to understand equiprobability in a more complex situation and additive probabilities. Others, such as Caleb, Maya, Alex, and Josh, had mechanistic or deterministic reasoning which hindered the other probabilistic ideas involved in this task.  This interpretation was also supported by the findings of Fischbein et al. (1967).  The researchers found that the older children in their study provided poorer responses as they mostly tended to focus on the deterministic interpretation of the phenomena.  Similarly, those students often used mechanical-geometric principles in order to justify their choice of certain paths.

**Pre-Interview Task 2: Ice-Cream**




Jan's Snack Shop has 3 flavors of ice cream: vanilla, chocolate, and strawberry.  The ice cream can be served in a dish, a sugar cone, or a regular cone.

There are 9 people who choose 1 dip of ice cream in a dish, or in a sugar cone, or in a regular cone, and all of their choices are different. List or show the 9 different choices.

Could another person have a choice that is different from one of these 9 choices? Why or why not?

In this 2003 NAEP task, students are expected to list nine different possible combinations of three flavors of ice cream served in three different kinds of containers. Moreover, the task requires students to recognize the sample space which is all possible combinations of having one dip of ice cream in a container.  In this study, the majority of students showed all nine different choices by using different methods which allowed them to justify that there were no other possible different choice than these nine.  Those who had incomplete list of choices asked in the task either did not have a systematic way to find the choices or included choices that were not included in the task, but based on her experiences.

Alicia started with listing the first three choices (see Figure 11). When she said "two strawberry in a dish," I reminded her that each person could choose only one dip of ice cream in this task:

> *Alicia*: Oh! So you are saying like three people could do this one [vanilla-regular], three people could do that one [strawberry-dish], and three people could do this [chocolate-sugar cone]
> *Sibel*: But they have to do…Every person has to choose different.
> *Alicia*: Hmm. Ohh! I get it I get it. So you could do another strawberry but in a regular…

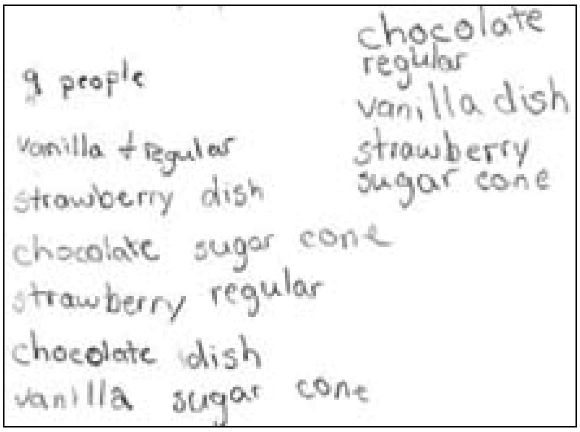



Figure 11. Alicia's response in Pre-Interview Task 2: Ice-Cream.

Then, Alicia went on to complete her list of all different nine choices. She knew that there were nine possibilities she was supposed to find because of how the question was stated in the NEAP task. With the prompts for what was stated and asked in the problem during the interview, she finally came up with a strategy to generate her list. Each time she tried to put a different dip of ice cream in each container by checking what has been already listed, such as "I did vanilla and sugar cone, so I can do chocolate and sugar cone." When I asked her the second question in the task, "Could another person have a choice that is different from one of these 9 choices?" she responded quickly by saying, "I don’t think so because there is vanilla in regular, sugar cone, and in a dish, but strawberry has a dish, regular, and in a sugar cone, and chocolate has sugar cone, a dish, and regular.” She then added that those were the only possibilities if people could only choose one dip of ice cream. In my opinion, once Alicia developed a systematic way to generate all nine different choices, she already knew that it was impossible to have another difference choice. Therefore, she reasoned with the list she had.

Similarly, Josh responded to the first part of the task by listing the possible choices, but his was a more compact display (see Figure 12). When he listed vanilla, chocolate, and strawberry and then dish, sugar cone, regular cone next to each ice cream, he knew that he had all nine different choices asked in the task and that there were no other possible different choice.

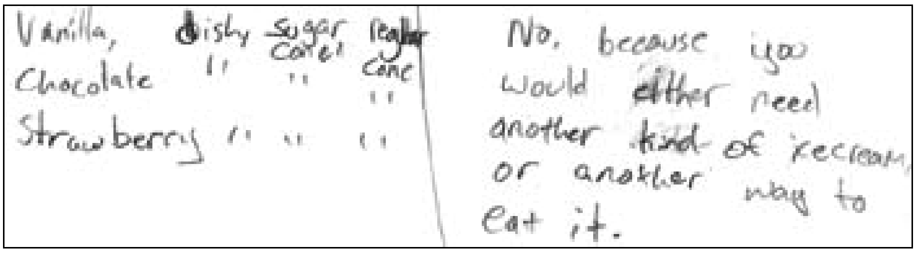



Figure 12. Josh’s response in Pre-Interview Task 2: Ice-Cream.

Caleb’s solution to the task involved a pictorial representation of all nine different choices (see Figure 13). First, he showed nine choices without a systematic method and ended up with some choices as the same as others. Then, he started with three different containers, such as dish, sugar cone, and regular cone, and put different ice cream in each, and then rotated each ice cream flavor from dish to sugar cone to regular cone. After that, he was sure that these nine different choices were the only possible ones “unless you mix sugar cone with the dish, but that’s the only thing.”

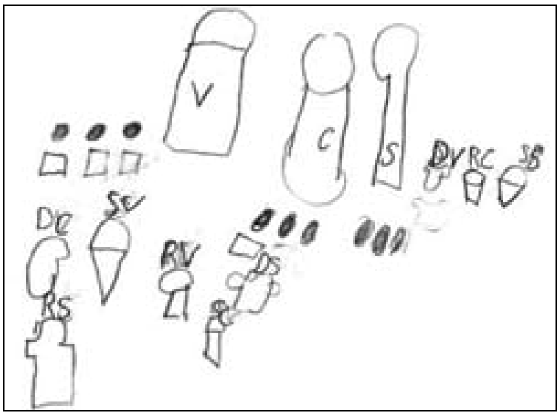


Figure 13. Caleb’s response in Pre-Interview Task 2: Ice-Cream.

Emily also was able to show all nine different choices. She began to count using her fingers as she looked at the picture of choices in the task and matched the ice cream flavors with the containers: “vanilla in a dish, vanilla in a sugar cone, vanilla in a regular cone, chocolate in a dish, chocolate in a sugar cone,…[kept counting but not verbally stated the choices].” Then, she showed all nine possibilities by using paths that linked the ice cream flavor with the different kinds of containers (see Figure 14). That was the justification she used for the second part of the task: “That’s all the choices there pretty much is unless they have a hidden stash of different ice cream.”

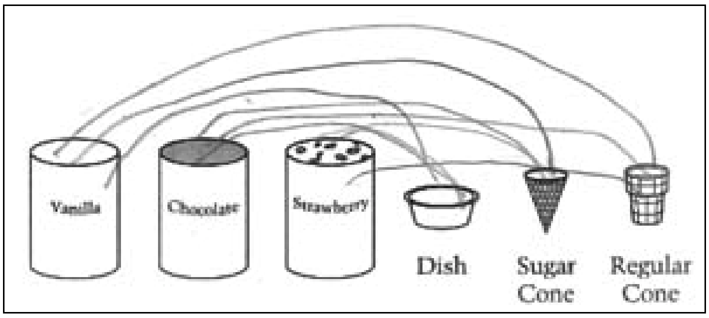


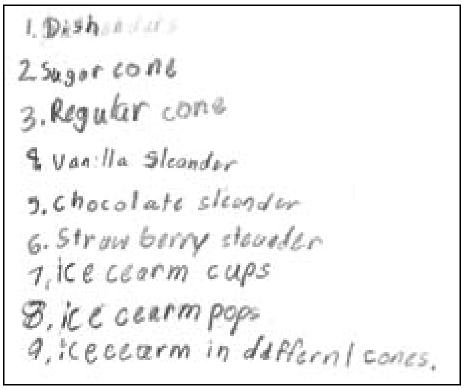

Figure 14. Emily’s response in Pre-Interview Task 2: Ice-Cream.

With Alex, I needed to prompt him to what was given and asked in the task since he did not know what to do initially.

*Alex*: I don't know.
*Sibel*: What kind of ice creams do you have to choose?
*Alex*: Vanilla, chocolate, and strawberry.
*Sibel*: Okay. And those can be served in?
*Alex*: A dish, or sugar cone, or regular cone.
*Sibel*: Okay. And each person can choose only one dip of ice cream in one of these [pointing to the picture], right?
*Alex*: Yes.
*Sibel*: And so what do you think a person could get? Let's say if you were to choose and you were one of these nine people.
*Alex*: I choose strawberry in a sugar cone.
*Sibel*: Okay. What would be the other choice?
*Alex*: Chocolate in a dish.
*Sibel*: Huh-uh. So can you keep doing this? What else could you get?
*Alex*: Vanilla in a regular cone.
*Sibel*: Okay. Any different one? Any other one?
*Alex*: No.
*Sibel*: No?
*Alex*: Yes. You could put chocolate in a different cone. Then in a regular cone. You can put them in a sugar cone and strawberry in a dish and vanilla in a regular cone.

I encouraged him to write down or show the different choices he just said on the paper, but he said he did not know how to write it. Then, I asked him again if there were another different choice than the ones he already stated, but he did not have any other possible choice of ice cream. Since he did not have a systematic way to show all possible combinations of three flavors of ice cream served in three different types of containers, he was not able to recognize all the possibilities asked in the task.

Maya initially did not recognize the sample space stated in this task. Therefore, she started listing all six choices given in the problem (see Figure 15) and then added some other choices, which were not part of the problem but she has seen before, simply to complete the list of nine different choices asked. However, at the end of her list, she mentioned having "ice cream in different cones." When I asked her explain what she meant by that, she said that they could have vanilla in a dish, chocolate in a sugar cone, and strawberry in a regular cone.  She also thought “they could have just three strawberries in three different cones,” but she still could not list all possibilities in a systematic way.  Furthermore, failing to understand the sample space in the task led her respond to the second part incorrectly as well.  She simply came up with various ways to have an ice cream assuming that there were other choices, such as having strawberry or chocolate ice cream with sprinkles.



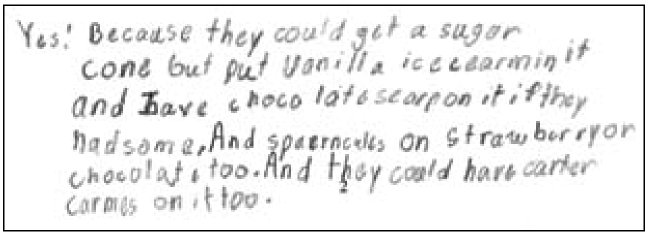




Figure 15. Maya’s response in Pre-Interview Task 2: Ice-Cream.

In sum, prior to the teaching experiment, Alicia, Emily, Caleb, and Josh were able to identify all possible combinations of having one dip of ice cream in a container. Moreover, their distinct and systematic approaches to show all nine different choices led them justify that no other different choice than these nine was possible.  Since Maya and Alex did not develop a systematic way to find all possible combinations in the task, they had difficulty generating all possible choices.  Therefore, Alex thought that the six choices he verbally stated were the only possible ones.  However, Maya believed that there were possibly more than nine choices since she tended to include the choices not in the task, but based on her experiences.

## Pre-Interview Task 3: Swim Team

There are 3 fifth graders and 2 sixth graders on the swim team.
Everyone's name is put in a hat and the captain is chosen by picking one name. What are the chances that the captain will be a fifth grader?
A) l out of 5
B) l out of 3
C) 3 out of 5
D) 2 out of 3
Explain how you got that answer.

In this 1996 NAEP task, the participants were asked to find the probability of choosing a fifth grader as the captain of the swim team and to explain how they got the answer, which was not part of the original NAEP item. Students needed to represent the probability as the ratio of the number of favorable cases (3 fifth graders) to the number of all possible outcomes (5 students in total) (Classical view of probability).

Emily, Josh, and Alex responded that the probability was 3 out of 5 with the following explanations:

*Emily*: Because two plus three is five and there is three 5$^{th}$ graders.
*Josh*: Because there are three 5$^{th}$ graders and three plus two is five. So the chances are 3 out of 5.
*Alex*: There is more 5$^{th}$ grader than 6$^{th}$ grader.

The explanations provided by Emily and Josh indicated that these two students were able to quantify the probability of an event according to the definition of the classical view of probability. In other words, they were able to make a relationship between the part and the whole as Piaget and Inhelder (1975) would argue. However, Alex's explanation was not sufficient to conclude that he referred to the relationship of the part to the whole.

Both Alicia and Caleb believed that the chance of having a fifth grader as the captain was "2 out of 3," but each provided different justifications for their answers when I asked them to explain how they got the answer:

*Alicia*: Because there are three $5^{th}$ graders and two $6^{th}$ graders. So it could be 2 out of 3.
*Caleb*: There are three $5^{th}$ graders and two $6^{th}$ graders. And $5^{th}$ grader has at least one bit more chance of getting picked.

Alicia simply used the numbers given in the problem intuitively without considering favorable cases and all possible cases. However, Caleb believed that "2 out of 3" represented the probability that a fifth grader would have one more (in an additive sense) chance of being chosen since the number of the fifth graders was one more than that of the sixth graders in the problem. In these examples, students seemed to make quantitative comparisons between the favorable cases (the number of $5^{th}$ graders) and the other cases (the number of sixth graders), rather than considering the relationship between the favorable cases and the whole (the number of $5^{th}$ and $6^{th}$ graders). Hence, they only considered the parts in quantification of probabilities.

Due to the focus on the parts only, Maya also incorrectly responded to this task. She chose "1 out of 3" simply because it was "more likely to be a $5^{th}$ grader and there are only three $5^{th}$ graders." She thought that a fifth grader was more likely to be chosen because there were more fifth graders and its probability was 1/3 since there were three of them. It seemed that Maya did not think of the whole when considering the favorable cases. Rather, she focused on the part of fifth graders as the whole.

To sum up, Emily, Josh, and Alex chose (C) "3 out of 5" as the probability of picking a fifth grader as a captain of the swim team in this multiple choice NAEP item. When asked to explain how they got their answers, only Emily and Josh justified their answers correctly by indicating the relationship between the part and the whole in quantifying the chance. When students, like Alicia (D), Caleb (D), and Maya (B), failed to recognize this part-whole relationship, they responded incorrectly. These findings were consistent with Piaget and Inhelder (1975) in a sense that part-whole reasoning was necessary to determine the probability of a single event.

**Pre-Interview Task 4: Stickers**

| *Stickers* | *Number* |
|---|---|
| Red | \|\|\| |
| Blue | \|\|\|\| |
| Yellow | \|\| |
| Green | ~~\|\|\|\|~~ \|\| |

The 16 stickers listed above are placed in a box. If one sticker is drawn from the box, which color is it most likely to be?

A) Red
B) Blue
C) Yellow
D) Green

Explain how you got that answer.

In this 1990 NAEP task, students were asked to identify the most likely outcome based on the information given in the table. It required students to compare the likelihoods of possible outcomes based on the frequency of each outcome. I also included the part with the explanation which was not in the original NAEP task. All participants in this study responded that the green sticker was most likely to be picked because there were more green stickers in the box. Hence, they all had an intuitive idea of probability as a function of the given frequencies (Piaget & Inhelder, 1975). Josh also further explained the effect of having more green stickers in the box with a notion of random mixture: “Because there are more green in the box. So, that way they could be more spread out. So, if you reach your hand in, no matter where you picked it, you know, the green is most likely going to be picked." Probably, Josh wanted to justify his response about the most likely outcome on the basis of frequencies.

### Pre-Interview Task 5: Marbles

In the following 1992 NAEP task, students needed to list the sample space by considering all possible outcomes of picking two marbles from a bag which consisted of yellow and blue marbles.

> Steve was asked to pick two marbles from a bag of yellow marbles and blue marbles. One possible result was one yellow marble first and one blue marble second. He wrote this result in the table below. List all of the other possible results that Steve could get.

| First Marble | Second Marble |
|---|---|
| y | b |
| | |

y stands for one yellow marble.
b stands for one blue marble.

Of the participants, three (Emily, Josh, and Caleb) were able to identify all possible ways to pick two marbles from a bag containing two different colors of marbles. A typical explanation for this response was, as Emily stated: "you can switch these (y and b) around, and he does not have to get two different colors of marbles, so he could get two yellows and two blues." The other students, however, had difficulty in generating all possible ways to get two different colors of marbles. For example, Alex and Alicia thought that the only other possibility was to get the blue first and yellow second. They did not think of a possibility of getting two marbles of the same color. Although Maya's response included "b and y" and "b and b" among several other repeated patterns of "b and y" and "y and b" (see Figure 16), they were generated in a nonsystematic way on the basis of different ways of picking marbles. For instance, she thought that Steve could put the marbles in a bowl and just pick them out without looking at the bowl, and then he would get “probably yellow first and then blue.” When I asked her whether there was another result Steve could get, she created a pattern on the far right of each column as a way of listing letters, b and y, but without any justification relevant to the context of the problem. One interpretation of the lack of identifying a sample space could be the difficulty of envisioning possible permutations for two elements, such as blue and yellow marbles.

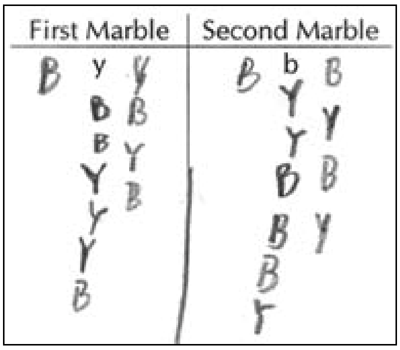
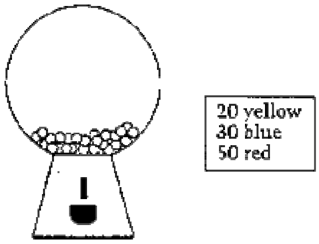



Figure 16. Maya’s response in Pre-Interview Task 5: Marbles.

In Task 5, Emily, Josh, and Caleb came into the teaching experiment with an understanding of possible permutations for two elements. However, Alicia, Alex, and Maya had difficulty in generating all possible outcomes of blue and yellow marbles in this task.

### Pre-Interview Task 6: Gumballs

In the following 1996 NAEP task, students needed to make a prediction about the number of red gumballs if 10 gumballs were picked from the gumball machine. Given that the gumballs are well mixed and the half of them are red, one would expect to get 5 red gumballs by using a proportional reasoning.




The gum ball machine has 100 gum balls; 20 are yellow, 30 are blue, and 50 are red. The gum balls are well mixed inside the machine.
Jenny gets 10 gum balls from this machine.
What is your best prediction of the number that will be red?
Answer:____________________ gum balls
Explain why you chose this number.

Of those who predicted 5 red gumballs, two (Caleb and Josh) provided an explanation based on proportional reasoning. Both indicated 5 being half of 10:

*Caleb*: Since this [the number of red gumballs in the mixture] is half of that [the mixture in the gumball machine] and if she got 10, she could have like 5 red gumballs, and it could be 2 yellow and 3 blues.

*Josh*: Because if you were to take all the zeros off because there are 100 gumballs in total, 5 would be half. So, um, and because 20 and 30 are both, 2 and 3 are both might got 5 and 5 and 5 are both halves.

It was evident that Caleb made his prediction in proportion to the population of 100 gumballs. It could be also argued that Josh's strategy "take all the zeros off" indicated a proportional reasoning when he talked about 5 being half of 10, like 50 being half of 100. Furthermore, the other two students (Alica and Alex) who predicted 5 red gumballs supported their responses by noting that there were more red gumballs in the mixture. With the same explanation, Emily and Maya made different predictions: "6" and "probably 7of them", respectively.

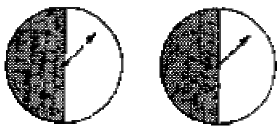
*Emily*: Well, most of the gumballs are red, so there is probably going to be just a little bit more red gumballs than all of the other color gumballs.

*Maya*: Because there is more reds than the others. So there are probably be more..she would probably get more reds than the other colors.

Both Emily and Maya thought that with more red gumballs, there would be better chance that Jenny would get slightly more red gumballs than the other color gumballs. Note that Alicia, Alex, Emily, and Maya were not able to provide any justification beyond "there are more red gumballs," and thus there was no evidence of using proportional reasoning in this task.

In summary, Caleb, Josh, Alicia, and Alex responded that five gumballs would be red. While only Caleb's reasoning was in proportion to the population, Josh used "take all the zeros off" strategy and Alicia and Alex simply reasoned with having more red gumballs in the mixture. Emily and Maya used the same reasoning as Alicia and Alex, but they predicted "6" and "7" respectively to indicate a better chance of getting red gumballs.

**Pre-Interview Task 7: Spinners**



> The two fair spinners shown above are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once. James thinks he has a 50-50 chance of winning. Do you agree? Justify your answer.

In this 1996 NAEP task, students needed to determine the probability of winning a game played with a pair of spinner. Since each spinner was half black (b) and half white (w), the sample space consists of the following outcomes: bb, ww, bw, and wb. Given all the possible outcomes, one would expect that the chance of winning the game (i.e. having both arrows on black) is 1/4, rather than "50-50." One could also reason with the multiplication principle (i.e., 1/2 * 1/2 = 1/4, probability of compound events) since each spin is independent of each other.

The student responses varied in this task: “No” (2), “Not exactly” (2), and “Yes” (2). For example, Alicia and Alex did not think that James had a 50-50 chance of winning for the following reasons:

> *Alicia*: Maybe if he spun it, he could have got like. Hmm. He could have got one on white and one on black. So, I am not sure if he has a 50-50 chance of winning.
> *Alex*: If the arrows start on white, it would end up on the same place. Then he couldn’t win.

Both Alicia and Alex did not agree with James, but with incorrect reasoning. While Alicia’s explanation seemed like a simple guess with no probabilistic reasoning, Alex’s reasoning was causal rather than probabilistic. Maya and Caleb tended to disagree with James also when they replied “not exactly” and justified their responses based on causal reasoning. For instance, Maya thought that James would be “close to win” because “here on the picture, the arrows are on the white. And if he spins it, then it would go back really close where it was.” Moreover, Caleb believed that “like 50-50 chance of winning, but sometimes 75-25 chance of winning and just depends on how well you spin and stuff” and thus James might not win the game. Unlike these four students, Josh and Emily, thought that there was 50-50 chance of winning the game. Then, they justified their responses with equal areas of white and black on each spinner (i.e., “each one is 50-50”; “even number of black and white on each side”).

In brief, the participants did not reason with either the sample space or the multiplication principle in the pre-interviews even though the responses of some of these students in the previous tasks, such as Tasks 1, 2, and 5, suggested that they had some conceptions of sample space and multiplication principle in other contexts. While Alicia simply guessed with no probabilistic reasoning, Alex, Maya, and Caleb responded based on causal reasoning. Moreover, Emily and Josh reasoned with equal areas of white and black on each spinner when they thought the chance of winning the game was 50-50. In the literature, the study of Cañizares, Batanero, Serrano, and Ortiz (2003) with children of ages 10-14 suggested that failing to recognize the fairness of a game in a compound event was due to the difficulty in differentiating equiprobable and non-equiprobable events due to the equiprobability bias (Lecoutre, 2002; see Chapter 2). Hence, in this study, students who reasoned with "50-50" chances in each independent event did not possibly establish whether the compound events were also equiprobable or not.

**Summary**

In the preceding subsections, I described the participants' understanding of probability concepts and their reasoning for each interview task. In general, the findings suggested that students mostly utilized deterministic and mechanical reasoning to determine the channels with equiprobable routes to exit 1 and exit 2 in the "Channels" task. Therefore, many of the students could not correctly identify the channels with equiprobability. The majority of students developed a systematic way to find out all nine possible combinations in the "Ice-cream" task. When students were asked to determine the probability of an event as the ratio of the number of all favorable cases to the number of all possible outcomes (the "Swim Team" task), most of them lacked an understanding of the part-whole relationship. However, all of the students had a conception of probability as a function of frequencies (the "Stickers" task). In the "Marbles" task, only half of the students could find all possible permutations of two elements. Furthermore, most of the students estimated a sample in proportion to the population, but only two students explicitly provided an explanation with a proportional reasoning. Finally, none of the students were able to determine the probability of a compound event and their explanations often involved causal reasoning or the equiprobability bias.

As mentioned before, the purpose of the pre-interview with individual participants was to investigate their prior knowledge and reasoning before conducting the small-group teaching experiment sessions (Supporting Research Question 1: What are the students' prior knowledge about probabilistic concepts and probabilistic reasoning?). I also used the findings to form the two groups of students for the study. Therefore, I summarized each participant's conceptions and reasoning across the tasks in different probability situations prior to the teaching experiment study and quantified their responses in Table 6. I used a rubric given in Appendix B to score each participant's responses in the pre-interview tasks. In developing the rubric, I considered each correct response with a satisfactory explanation in every task and the NAEP scoring guide for the Tasks (2-7). According to their overall score, Emily, Alicia, and Alex formed Group 1 and Maya, Caleb, and Josh were in Group 2 so that the average score for each group was 10 out of 16 points. Moreover, the variety of each group's responses in the tasks seemed to be more or less equivalent in terms of correct/incorrect answers and different reasoning strategies used by each participant (see Table 5).

Table 5. The summary of each participant's responses and reasoning across the pre-interview tasks. ( * represents the correct answer)

| | Task1 (3 pts) | Task2 (2 pts) | Task3 (2 pts) | Task4 (2 pts) | Task5 (3 pts) | Task6 (2 pts) | Task7 (2 pts) | Total Pts (16) |
|---|---|---|---|---|---|---|---|---|
| **Alex** | None<br>Mechanistic reasoning<br>(0) | Not all possible combinations<br>(1) | C*<br>No part-whole relationship<br>(1) | D*<br>Probability as a function of frequencies<br>(2) | Incomplete<br>(1) | "5"<br>"More red gumballs"<br>(2) | "No"<br>Causal reasoning<br>(0) | (7) |
| **Alicia** | Figures A, D<br>Probabilistic reasoning<br>No addition of probabilities<br>(2) | All possible combinations<br>(2) | D<br>No part-whole relationship<br>(0) | D*<br>Probability as a function of frequencies<br>(2) | Incomplete<br>(1) | "5"<br>"More red gumballs"<br>(2) | "No"<br>Guess with no probabilistic reasoning<br>(0) | (9) |
| **Caleb** | Figures A, B<br>Mechanistic reasoning<br>(1) | All possible combinations<br>(2) | D<br>No part-whole relationship<br>(0) | D*<br>Probability as a function of frequencies<br>(2) | Possible permutations of two elements<br>(3) | "5"<br>Proportional reasoning<br>(2) | "Not exactly"<br>Causal reasoning<br>(0) | (10) |
| **Emily** | Figures A, B, D<br>Equiprobability<br>Multiplication and addition of probabilities<br>Probabilistic reasoning<br>(3) | All possible combinations<br>(2) | C*<br>Part-whole relationship<br>(2) | D*<br>Probability as a function of frequencies<br>(2) | Possible permutations of two elements<br>(3) | "6"<br>"More red gumballs"<br>(2) | "Yes"<br>Reasoning with equal areas (W-B)<br>(0) | (14) |
| **Josh** | Figure A<br>Deterministic and mechanistic reasoning<br>(1) | All possible combinations<br>(2) | C*<br>Part-whole relationship<br>(2) | D*<br>Probability as a function of frequencies<br>(2) | Possible permutations of two elements<br>(3) | "5"<br>Proportional reasoning<br>(2) | "Yes"<br>Reasoning with equal areas (W-B)<br>(0) | (12) |
| **Maya** | Figure A<br>Deterministic and perceptual reasoning<br>(1) | Not all possible combinations<br>(1) | B<br>No part-whole relationship<br>(0) | D*<br>Probability as a function of frequencies<br>(2) | Incomplete<br>(2) | "7"<br>"More red gumballs"<br>(2) | "Not exactly"<br>Causal reasoning<br>(0) | (8) |

# CHAPTER 6

# RETROSPECTIVE ANALYSIS OF THE TEACHING EXPERIMENT STUDY

In the preceding chapters (4-5), I described the design of the sequence of tasks used in the teaching experiment study based on the revised conjectures developed from the findings of the pilot study. Also, the results from the pre-interviews were presented to provide insights into each participant's initial knowledge about probability and probabilistic reasoning prior to the study. Next, the following sections document the findings within each of the tasks to address the main research question: How do students develop reasoning about distributions when engaging in explorations of chance situations through a sequence of tasks in which students were asked to provide predictions and explanations during the experiments and simulations with objects, a physical apparatus, and a computer environment?

## Task 1: Distributions in Different Settings

Like in the pilot study, the teaching experiment began with the task of discussing distributions of things in a variety of pictures, including animals in a field, leaves under a tree, and airline routes, which served as a starting point for reasoning about distributions. These pictures were used to let students think about aggregates, spread, and density, and reason about the ways things are distributed in the pictures (see Appendix D for the specific questions asked in this task).

The analysis of the first teaching episode brought forth three themes in students' reasoning about distributions in the pictures (the buffalo herd, the leaves, and the airline routes): (1) patterns, (2) density, and (3) causal and deterministic reasoning.

**(1) "Patterns":** In the pictures of both buffalo herd and leaves, students first noted the amount of buffalos and leaves by saying "lots of lots of buffalos" and "too many leaves." Their discussion of "manyness" in these pictures led them to talk about different *patterns*, particularly groups, chunks, and spread-out-ness. For example, students noted that there were many buffalos in the picture and the way they were distributed was not uniform, rather it was in different patterns, such as some in big chunks and some separate from those:

> *Maya*: There is lots of them.
> *Josh*: They are sort of a…They aren't all *packed in* one big formation. They are all sort of *spread out*. There is like I don't know, 20 of them that are *together* and there might be three of them that are *apart* from those 20.

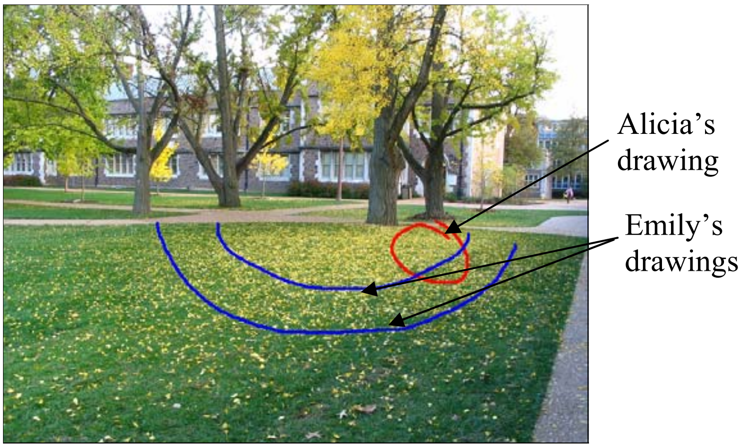



Figure 17. Alicia's and Emily's ways to display various groups of leaves under the tree.

When talking about the leaves under the tree, Alicia focused on the piles of leaves and wanted to circle them (see Figure 17), like they did in the buffalo picture. She added that they could count the buffalos since they had them in groups. Similarly, she was interested in groups as an attempt to figure out the amount of leaves. However, Emily thought that the leaves were distributed differently such that they were mostly around the tree trunk and some spread out further from it. Therefore, she suggested that they could show the groups of leaves by drawing semicircles around the tree trunk (see Figure 17). Later, both agreed that it was not possible to count the leaves, by using an analogy of sprinkles on a cake, but they could show where the majority and less of them were in the picture.

In the picture of non-stop flights from two cities (Birmingham and Dallas), unlike my conjecture relevant to this task discussed in Chapter 4, students did not necessarily focus on the distribution of destination cities from each departure city on the map. Rather, they tended to talk about distances from the original city to the destination cities. When asked about what they noticed about the non-stop flights on these two maps, they indicated shorter and further routes:

> *Alicia*: It depends…say it is all the way from here [Birmingham], you have to go *further distance*. Would you have to from here [Dallas], you have to go *shorter distances* because this one is shorter route and because this one is *close* all the cities you need to get, but this one is *further away* to all the cities you need to get to. So you have to take *longer distances*.
>
> …
>
> *Caleb*: They are all *close together* [Dallas map]…They [flights from Birmingham] are *longer* than this [Dallas map].

Later, Alicia continued to reason about the distribution of non-stop flight in terms distances and time, such as "Maybe they [the arrangements of flights in the two maps] try to show us if you have two cities…it would take a longer time to get to the cities you want to if you are in a further distance, or closer if you are in a closer distance." Moreover, some groups of destination cities, particularly with respect to their locations on the US map, and their "spread-out-ness" were noted in the discussions:

*Josh*: There is a lot of them in the middle of the US. There is like four of them that are all in that area. And then there are lots of them in the south. And you can also say a lot of them in east [he is partitioning the map].

*Sibel*: How about the other one, the one below?

*Josh*: In that one, it is basically they are all in south. There is nothing up there. You can't go anywhere up there [above the line he drew]. Also the east is pretty empty. Also it is basically the middle of the US and it is also the further south. Basically right here [Dallas map], one, two, three,… there is twelve places you can go total. And up here [Birmingham map], there isn't many [counting]. Twelve. So there is basically the same number, but up here it is more *spread out* and down here it all *pushed down*. Like it is all below a certain line.

**(2) "Density":** Like in the pilot study, students began to talk about different patterns with the notion of density. For example, some patterns were described as "close together", "in big chunks", and "packed together", particularly in the buffalo herd picture. Also, as illustrated in the following excerpts, students explained the places where they saw more and less of the buffalos on the picture with the more and less density areas:

*Josh*: Up here there is not that many. They are *more spread out*. But in here they all are *packed up tightly* and here there are right up here *a little tight*.

...

*Sibel*: How do you know there are less there?

*Emily*: Because it looks like it. If you look at the groups, it looks like there is a lot less over here [left side of the diagonal] than over here. Because the groups are *a lot bigger* over here [right side of the diagonal] [Alicia agreed with her]

*Sibel*: Where do you see more of the buffalos? Can you show us please?

*Alex*: This half. More. [right side of the diagonal]…because it is *more thick with buffalo*.

When I asked them whether they could guess how many buffalos there were in the picture, students in Group 1 tended to count the buffalos in the groups they circled on the picture whereas in Group 2 where more discussion about density took place. Josh developed a strategy to estimate the number of buffalos by partitioning the picture into six about-equal-sized areas. He started with an estimate of 30 buffalos in the lower right corner and made the other estimations by comparing the crowdedness in each partition with his first estimate. In the partitions where he guessed 35 or more buffalos, Josh gave explanations including "all pushed in" and "they are more packed together" while in his estimate of 20 buffalos, he said "not that many because they are spread-out." Similar to the results in the pilot study, estimating the number of buffalos by looking at the patterns of the high and less density regions entailed an understanding of density because in order to reason about those numbers, Josh needed to contrast each area with respect to its crowdedness.

The pattern of leaves falling off the tree onto the ground was visible to the students in a different way than that of buffalos. Like Emily's response in Figure 17, Maya showed the regions where the amount of leaves got less and less the farther they were from the tree trunk. That led to a discussion about the areas with high and low density. In the following exchange, for instance, the color scheme was interpreted as an indication of density. In other words, the more yellowish areas were high density (because the leaves were yellow in this picture) and the more greenish places were less density as the leaves were more scattered on the grassy area. See the excerpt below.

*Maya*: This is lots, lots, lots, less, less [drawing semicircles on the picture]. It's like lots, lots, lots, less, less, and then like less [another drawing]. Like it keeps getting like [pause][*S: less and less?*] Less, yeah. It keeps getting less and less and less.

*Sibel*: How do you know there are more here?

*Maya*: If you look at the leaves here, they are…these are green. This is the same color as these. And these leaves over here (right under the tree), they are like the same color.

*Josh*: Basically you can see that in this area, it is a *stronger* yellow. But in starting from here it starts to turn to *dark* green, which is basically the grass under it. Right up here it is *strong* yellow.

**(3) "Causal/deterministic reasoning":** When students were asked to explain the various patterns noted in the pictures, their responses often included deterministic and causal interpretations. For example, in the following excerpts, students were discussing what they noticed in the pictures of leaves and airline routes and how the buffalos might be in the groups that they determined on the picture:

> *Alex*: [on the leaves picture] There is less on the edges because the tree is further away from it.
>
> *Emily*: I think that's pretty true because the branches don't go out, say like as much as this tree [the tree behind it]. So not many leaves would be here [further] as right beneath the tree.
>
> …
>
> *Alicia*: [on the Birmingham map] I see more they are going to east side than they are going to west side. Maybe it is because if they were going to a vacation and they were to go somewhere close and they have like only three weeks and these take three weeks to get there [north of Birmingham], then there might be more coming over here. It takes long to come over here [west of Birmingham] so because these are longer distances than they are over here.
>
> *Emily*: [on the Dallas map] I think more of them are going to the west side maybe because maybe more major cities are closer on the west side.
>
> …
>
> *Maya*: [on the buffalo picture] I think they form a giant circle that all the strong ones go around the circle and the second strongest and they have all the young ones in the middle so that they could protect the young once.

When students tended to make deterministic and causal interpretations, in order to elaborate on that they were asked, "Does it matter how they are organized?" Their responses indicated different beliefs about when it matters and when does not. For example, Josh believed that it would matter how the buffalos were organized, such as having the strongest buffalos around the weaker and younger ones, if there were predators in the area "because if they are all spread out, once can be swiped away and no one notices." Josh however thought that the way the leaves were distributed under the tree "sort of matters" and then added "but it wouldn't change out lives." Similarly, other students did not think that it would matter for the leaves because they would just naturally fall wherever. Those included:

*Emily*: Probably not because if you were making pattern of these, then it would matter how it's organized. But these are just falling, it doesn't matter because leaves aren't really alive.

…

*Maya*: I don't think very so because they are going to probably wreck the leaves anyway. Somebody is probably gonna come and wreck all of these leaves up. Probably get rid of them. But I don't think it really matters where it falls. It is probably gonna fall wherever it falls. A leaf [drawing on the picture] could fall off the tree and then the wind can blow it like all the way over here or something [drawing on the picture]. It really doesn't matter. It probably go wherever the wind takes it.

Overall, this task was indeed a useful starting point for looking at students' informal language to talk about distributions and their qualitative/quantitative reasoning about these distributions as it built upon what students already knew. Mostly students reasoned about the distributions by noticing groups, particularly in the pictures of the buffalo herd and leaves. This then led them to discuss the higher and lesser density areas by comparing different group patterns. Similar to the results from the pilot study, their explanations for such patterns were mainly driven by deterministic and causal interpretations of the context.

## Task 2: Dropping Chips Experiment

Following students' qualitative and quantitative reasoning about the distributions in different pictures, including the buffalo herd, the leaves under the tree, and the airline routes, I asked the students to conduct a sequence of activities to investigate distributions of objects in designed settings. In these tasks, students were asked to predict, generate, and interpret distributions of objects (i.e. chips) when conducting various experiments with some variations (see Appendix D).

*Experiment 1: Dropping 20 chips through a tube when the tube was <u>15 inches</u> above the ground.* Students' initial predictions and explanations for their responses included themes around "middle chunk" (mostly in the middle and closer), "some spread" (expectation of variability), and "deterministic explanations" (the way the chips land on the ground focusing on the physical aspects of it).

All students expected that some of the chips would be in the middle, close to the dot and the rest would spread out on the sheet, rather than landing "in perfect little stack" or staying "like stacked up on the dot." When asked to mark where they thought most chips would land on the sheet, students in Group 1 marked a distance from the dot by which they expected most chips would fall (see Figure 18) whereas students in Group 2 drew a full circle around the dot. Their responses indicated that the majority of the chips would be in the middle and closer to the dot (high density area) with some possible spread, with an explanation like, "most likely they won't all land on the same spot because they are going to knock each other" as Josh stated. In Group 2, students also quantified their predictions of the number of chips that might land beyond the regions they marked on the sheet. For example, Josh said, "Out of twenty maybe, two or three" and Maya thought "four or three."

Figure 18. Students' predictions for the distribution of chips (15" above the dot) in G 1.

Students' initial reactions to the results of their first dropping chips experiment, in comparison to their predictions, included discussions of "majority," "the chips beyond the predicted region," and "symmetry around the dot" with some causal explanations on the basis of their observations of the phenomenon. For example, Alicia noted the places where the most of and none of the chips landed and provided an explanation using the idea of being closer to the middle (the dot). Moreover, students were surprised by the results being more spread out beyond the region they predicted. The dialogues are presented below:

*Alicia*: There was like none over here. I was wondering because *most* of them landed around here [the dot] and *none* landed around here.
*Sibel*: Why do you think it happened that way?
*Alicia*: Maybe because the dot is more over here than it is over here…It is a *shorter distance* from around here than it is from around here to the dot.
…
*Caleb*: That's a lot more than I expected…That much being outside.
*Maya*: It's probably because they piled up, then spread out. Like the Frisbee. If you throw whole bunch on the same place, they are all scattered across because they are all bumping into each other.

Furthermore, Josh pointed out that the middle chunk was not perfectly landed around the dot, instead they were "pushed off" one way (see Figure 19). It seemed that when students marked a circle around the dot, they tended to expect a uniform and symmetric distribution around the dot, rather than most concentrating in a particular area in that region.

*Josh*: I thought that most of the chips were added up. You know scatter between there, but it doesn't look like. It looks like most of them were *pushed off that way* [towards Caleb]. And it looks like last few came in this area.
*Sibel*: Why do you think it happened that way?
*Josh*: Because I mean. It looks like you know the first part were all *pushed up that way*, but if one were to come down and hit right there, all these would *push out further* and it would shut over here.

Then, he offered a conjecture about the way the chips landed based on what he observed in this particular experiment result: “These, right here, look like they were the first ones to fall because they are right smacked up in the middle. And all of the other ones seemed to be pushed out to the sides.”

Figure 19. The results of first dropping chips experiment (15” above the ground) in G 2.

After the discussion of the results from the first trial, students made their predictions for another trial with the chips by marking the region in which they expected most of the chips with a different color marker. Their responses revealed that the next distribution of chips would be similar to the previous one in a sense that most of them would be in the middle, closer to the dot, but the rest might spread out in different directions this time. They were able to see the middle chunk as a common property now, but still expected some variation in its density around the dot from one trial to another. Sometimes that variation was referred as “spread out” like in Josh’s response below.

> *Emily*: Like maybe some more might go over here or less like. Like, it is similar to the other one, but not the same.
>
> …
>
> *Alicia*: Maybe some might land, maybe more might land out there this time. And less might land here and some might land over here.
>
> …
>
> *Josh*: It won’t be the exact same. The chips won’t be in the exact same place because like, if you drop, there is one there, one there, one there. And I am

going to drop them again. They are all *spread out*. So it depends. They are not always going to go in the same spot.

When Josh and Maya showed the region in which they expected most of the chips for the next trial, Caleb thought that the area was not big enough. Then, he wanted to mark a region in which there would be as many chips as outside because he believed that it would be "fair" to have "even" number of chips both inside and outside of the circle. The dialogue is presented below.

*Caleb*: The first ones would be probably up there [around the dot] after a while it is going to get too large, then it is going to have to fall sometime, then it is going to spread out all over.
*Sibel*: Where all over?
*Caleb*: Outside of the region.
*Sibel*: Oh, outside of the region. Do you expect more outside or less?
*Caleb*: *Even*.
*Sibel*: Even, like as many as the ones inside?
*Caleb*: Yes.
*Sibel*: Why?
*Caleb*: It seems to be *fair*.

Hence, he drew a slightly bigger circle which was about an inch further than the region that Maya and Josh predicted around the dot. When they conducted the experiment, Josh noticed that there were three chips outside of the middle chunk: "See some of them are way out here." Then, Caleb said that he was not actually right with the following explanation: "because I thought like most of them would be out here [outside of the middle region]. It would be 50-50, but it is not." It seemed that Caleb should have predicted a smaller region around the dot in order to have as many chips inside as outside since dropping the chips at this height would mostly produce a big middle chunk closer to the dot and some scattering outside of that middle.

Furthermore, other students provided causal and physical explanations when interpreting the differences between the results and their predictions for the chips. For instance, Alicia argued, “Maybe different people did it and different people have a way of dropping them. I sort of did them like that [showing the way she dropped chips] and some dropped later than others. And he (Alex) just dropped them all at one time.” Then, I asked Alex whether he did something differently when dropping the chips. He responded that he dropped them faster. Alicia, then, claimed that when he dropped the chips faster, “they all just went there [the middle area] and then the last remaining somewhat spread outs.”

*Experiment 2: Dropping 20 chips through a tube when the tube was <u>30 inches</u> above the ground.* When students were asked to make predictions about the second experiment in which the height was doubled, all students expected that the chips would spread out more outside, with less of them landing on the dot. Some of the responses were as follows:

> *Alicia*: Oh, yeah. Because if this one (the tube) is sort of higher than the other one, so maybe when you drop them, they all spread out.
> …
> *Alex*: I think more of them will land out…Out in the white area.
> …
> *Alicia*: It is not as big as chances that they will land on the dot than any other places. So most of them…out than on the dot because the tube is way up here and in the other one that’s kind of down there.

Apparently, students expected different result than those in the previous experiments, and provided various explanations. Specifically, after watching the chips landing on the floor in the previous experiment, their response mostly involved causal explanations based on considerations of the physical aspect of dropping chips on the floor, such as “harder landing” and “jump and bounce out,” for example:

> *Emily*: I think more of them will go out because they have a harder landing so they might shattered somewhere over here.
> …

*Caleb*: This time I would think it is gonna be outside because the higher it is, the more likely it is gonna jump and bounce out off the region.
*Maya*: I think since it being higher, I think probably it is going to go straight for a long time. Probably it is gonna go straight and turn or something. [She draws a medium size circle in the middle and she thinks that a bunch of them will land in that region and some will bounce off and go further.]

After students conducted the experiment in each group, they considered their predictions as "right" because the chips did "scatter out more." In order to find out whether students thought the way the chips spread out was random or systematic, I asked follow-up questions on what they meant by saying "spread out." For example, Alicia responded, "I think maybe they just go *wherever*, they *don't go to a particular spot*. They *just go everywhere* instead of just landing right in the middle." Alicia seemed to expect a random spread, rather than chips systematically going to "a particular spot." Moreover, students continued to provide explanations based on the physical aspect of the experiment. For example, Emily conjectured that the chips dropped from a higher position would fall "faster so it would have more time to get faster and faster and finally it hits the ground" and then scatter out more.

Maya began to see similarities between the distribution of chips in this experiment and the distribution of buffalos in the first task. In her interpretation of the results, Maya showed the different groups of chips as patterns on the sheet.

Maya: I noticed that I was right too. It was gonna go over here and then that was gonna pop out and jump every. I notice that they go into groups [Groups of two and three].
Sibel: What about those groups?
Maya: They are like the buffalos. They are in groups see. [Showing them on the sheet]
Sibel: What else do you think about these results and your predictions?
Maya: *The higher it is, the more it goes*. The more higher it is, the more it is spread out [showing by dropping a marker].

As seen in the excerpt above, Maya's last lines offered an example of making generalizations from the previous results: The higher it is, the more it is spread-out. Maya also added that "it is probably getting bigger because maybe the way we put it, like let go them. Like I hold at least five of them at one time and I like going them all." She thought that if all chips were dropped at the same time, they would all spread out without being in groups. Then, Josh made a conjecture about how the chips were dropped and their distribution on the sheet: "If you let them all go at the same time in the same place, they are gonna spread out. But if I were to stick them together, you know because they have been stuck together for a long time in someone's hand, then I were to drop them, they are right next to each other. And then I drop the next group, it sort of spreads out a little bit more. Then I drop the next group, it spreads out even more." Maya and Josh also thought that when Josh was holding the tube, he moved it because his hands were shaking. As they speculated more and more about how the results happened that way, I decided to ask them, "Does it matter which directions would they go?" I got differing responses. For example, Maya believed that it would not matter where it landed "unless it has something very important to do with your life." However, Josh thought it would matter when one considered "the laws of science", such as gravity, which led chips to fall straight down "unless you are putting force on that to move that way [right] or that way [left]." Similar to the responses in the Task 1 (i.e. whether it matters how the buffalos and the leaves are distributed), the responses in the dropping chips activity varied. Again, for Maya the consequences of the phenomenon would determine whether it matters or not, whereas for Josh it was the external forces.

After the discussion of the results from the experiments with variation in the height of dropping chips, I asked students to make predictions for if they were to drop 30 chips instead of 20 at both heights (15 and 30 inches up). Maya thought that it would give similar results in terms of the shape, but the amount of chips in the regions they predicted would increase as there were more chips to drop. Caleb who had a similar idea made it even more specific and claimed that "there will be twice as much, but still be like in the same places [like inside or outside of the region they predicted]." It seemed that Caleb lacked proportional reasoning when he predicted "twice as much" chips in the same regions when the number of chips was increased from 20 to 30, rather than doubled. Moreover, based on the previous experiment results, Josh made a generalization: "if there are more, they will spread out. If there are less, they will land closer. If there are equal amount of chips, they will land on about the same area." He thought that the chips would spread out more if there were more of them to drop and so did Alicia and Emily. However, Alex expected that more chips would land "closer to the middle" because he thought that there were more in the middle than outside in the previous experiments and "they are heavier, so they push down on each other." The "heaviness" aspect of having more chips to drop led him also to think that the chips would scatter farther out when dropped from 30 inches up: "Because they are heavier and they go faster when they are coming down."

As I planned in my task sequence, at this point I wanted to introduce the statistical term "distribution" to the students to talk about the results of chips on the floor and the next examples of various distributions. When I asked them if they had heard the term "distribution" before, some of them already had an idea about what "to distribute" means.

Those ideas included "to hand out," "to give out," "to spread out," and "to add to something." Then, I explained how we used "distribution" in mathematics to refer to the way in which objects spread over a space or an area and suggested that we could start using that term from now on.

When students were asked to compare and contrast all the distributions they generated with the chips, they noted a general feature, "the middle chunk". They also noted the differences from experiment to experiment caused by the variations in the height and the number of chips in each case. Then, I asked them whether they could think of any types of distributions which did not have a middle chunk (or no centered distribution), and they came up with examples of either a skewed distribution of chips or a uniform distribution of other objects. For instance, Maya and Josh demonstrated an experiment with the chips and the tube in which they held the tube with an angle, rather than perpendicular to the floor and shot the chips through it. They argued that the middle of the distribution did not have the majority of the chips anymore. One example of a uniform distribution was the game "Pickup sticks" as suggested by Josh: "At my house we just pick up the big bunch and spread them out. So, there really is no middle. It's just basically a million of sticks allover the place." Moreover, Alicia, Emily, and Alex mentioned their own experiences of dropping things, like a bunch of marker caps, pencils, markers, and the Lego pieces in the box, because they thought that those distributions did not have a "middle chunk", rather they "spread all over the floor."

## Task 3: Dart Game

In this subtask, students discussed various aspects of the dart game based on their own experiences. All students viewed the dart game as a "skill game" rather than a "chance game," which is also a common public perception. They said that the aim of the game was to hit in the middle to win (see Figure 20). Students in Group 1 argued that the middle had the highest point value because it was smaller than the other places. When asked whether one of those places was easier to hit than another one, Emily responded that each slice was the same size, but in the middle it was harder to get because "they sort of grow smaller." According to her explanation, the middle was the hardest to hit since it was both a smaller area and right in the middle. Moreover, Alicia believed that getting higher scores would depend on from where one throws the dart.


Figure 20. The dart board discussed in the Task 3.

When the same discussion occurred in Group 2, Josh's idea about harder places to hit was rather different in that he believed the slice for 20 points was "somewhere you would never hit" whereas the slices for 16, 7, and 19 were "most likely" and easier places for him to hit if he usually aimed at the middle. For Caleb, where to aim for was the most important aspect of getting higher scores because he believed that you would get higher scores if you aimed at the middle to throw the dot, i.e., "the closer you hit the middle, the more point you get. Because it is easier to hit, say 17, than the middle point." For Josh, however, the way of throwing the dart would determine whether you would get higher scores. For a strategy, Josh then added that he would aim for the slice for 4 points to try to hit near the center because of "the way the dart moves in the air."

### Task 4: Design Your Own Game

Considering the effect of height for the distribution of the chips, students created their own games in which they gave different points to the chips for landing near or far from a target. Before students started designing the sheet on which the chips would be dropped, they specified the height and the number of chips to be used (see Table 6). The students in Group 1 decided that each player should choose the height to drop the chips and initially they specified it as 55" or less. When I asked questions about their design, such as the choice of height and the number of chips, the regions, and the scoring, Alicia responded that each player would chose different heights "to get different results." I asked her to explain it a little bit more and she said "if it is lower, more over here [in the middle] and if it is higher, you will get less over here [in the middle]." They expected to get less point if the chips were dropped from a higher position, like 55", and Emily said "because we wanted to make it really hard."

Table 6. The specified rules for the designed games.

| | **Group 1: Game** | **Group 2: Game** |
|---|---|---|
| **Height:** | Between 21" and 55" | At 20" |
| **Number of chips:** | 20 or more at once | 3 chips at four rounds |
| **Final score:** | The total score | The total score |
| **Winner:** | Whoever gets the most point | Whoever gets the most point |

Once they started playing the game, Alicia as the first player chose 40” and then Emily dropped the chips from 35” up. However, Alex wanted to choose a height as low as 15”. This led them to renegotiate what should be the lowest height one could choose to make the game “fair”, and they decided on 21” above the ground. The variety in their choices of height indicated that in order to win they should drop the chips from a lower position. Moreover, Alicia and Emily chose to drop 25 chips rather than 20 or 30 when they played the game, by drawing upon the discussion we had earlier in the *Dropping Chips* experiments. Their reasoning was that this was the way to get enough points to win the game since both girls earlier believed that if there were more chips, they would spread out more.

In the other group’s game, the height was fixed as seen in Table 6. Twenty inches above from the ground seemed to be “fair” for these students. For instance, Caleb said, “it seemed fair. Say for instance, 60 inches [he is demonstrating with the three chips], see it won’t be like fair score. It would be the lowest score you could get, like 16.” Then, he continued doing more experiments with three chips and they all agreed that if it were 1 inch above the ground, it would be easy and you would get 50 points every time because they would not separate. Moreover, Caleb thought that 15 inches was “a little better,” but “still not fair” whereas 20 inches would be “fair” since the chips spread out more. The students in this group also chose to drop a smaller number of chips (three chips at each round, a total of twelve chips) basically because it would be easy to add. However, Josh thought that dropping more chips might be “a bit even” because they would “spread out more so there would be a lot more 0, a lot of 1s and 3s, one 50.” This was another notion of fairness based on the effect of number of chips which was also consistent with Josh’s earlier belief, i.e., “if there are more, they will spread out.”

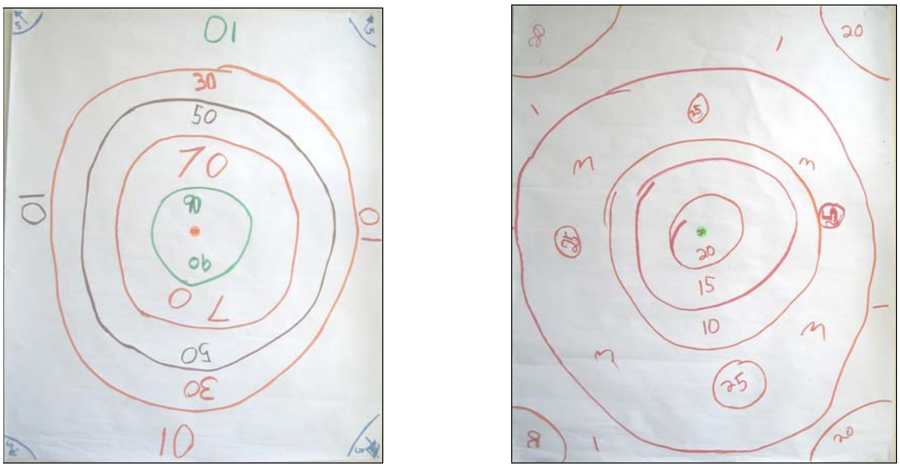



Figure 21. Student-generated games: (a) Group 1’s game (b) Group 2’s game.

While both groups designed a similar looking game sheet in terms of the concentric circles around the dot in the middle, which looked like a dart board, their games differed in the choices of the height and the number of chips dropped as well as in their scoring of the regions on the sheet (see Figure 21). For example, in Group 1’s game, the scores decreased by 20 points as it got further away from the middle and students assigned larger numbers to start with (i.e. 90 points vs. 20 points for the most inner circle in Figure 21 (a) and (b)). However, in the game designed by Group 2, the scores varied even in the same region after the 10-point area. For instance, in the 3-point region students created four (sort of symmetric with respect to the middle dot) areas to which they assigned 25 points because Josh said they were small. It seemed that the smaller areas were considered less likely places for chips to land on, similar to the findings of the pilot study. Also, both groups gave the highest score for landing on the dot (100 and 50 points in Groups 1 and 2 respectively) because it was the smallest area right under the tube. In addition, they often talked about how many chips landed right on the dot when they conducted experiments with the chips earlier. Even though the majority of the chips were close together in the middle, there were only a few on the dot, and fewer as they increased the height of dropping chips. Therefore, it made sense for students to give the highest score on the dot. As seen in the Figure 21, both groups had corner regions, but the students in Group 2 assigned 20 points on landing on those rather than 5 points in Group1's. Maya explained why they were worth of 20 points by saying "in case you put it [the tube] higher." The reason the Group 1 gave only 5 points (instead of a high point) to corner regions was that people would try to get those on purpose and this would make the game easy. Overall, to win in Group 1's game, one needed to have most of the chips closer to the middle and the chance of winning could increase if there were more on the middle dot. However, in the other group's game, one's chance of winning the game could increase as the chips spread out more and landed on the small areas in the 3-point region as well as on the corner regions worth as much as the middle chunk.

## Task 5: Gumballs Activity

To look at students' conceptions of chance in different contexts, I asked students to make predictions about the color of the gumball they might get from the mixture in the gumball machine. After the predictions, I mixed the gumballs in the machine and one student inserted a coin to get one gumball. Then, I put that gumball back into the mixture and the next student followed the same procedure. The common strategy to make a prediction in this task was to look at the bottom of the gumball machine to get it right.

For instance, students in Group 1 predicted that it would be yellow because there were more yellow gumballs closer to the bottom. When the result turned out to be red, Emily said “red could be hidden there [at the bottom].” Similarly, in Group 2, when the result was a blue gumball, Maya seemed to be surprised and said “there wasn’t any blue. Probably they were all at the bottom.” Even though their initial predictions, like their (theoretical) responses in the pre-interview task about the gumballs, were based on the proportion of the colors in the mixture, students tended to use deterministic reasoning in a real experiment situation, like they did in the previous task with the chips. In other words, probabilistic reasoning requires students to consider the proportion of the colors in the mixture whereas deterministic reasoning calls for making a plausible explanation when the prediction fails.

### Task 6: The Split-Box

During the fourth and fifth teaching episodes, the students worked on the *Split-Box* task in which they conducted various experiments with the “split-box” (Figure 22), an adaptation of the inclined box used by Piaget and Inhelder (1975). Similar to the findings of the pilot study, the students tended to look for patterns and to come up with some algorithm to predict the results of marbles in the split-box. Sometimes, they reasoned based on the previous outcomes and some physics, such as the possible paths the marbles follow and how fast they go. Next, I present these findings through various experiments that students in each group conducted.

Figure 22. The Split-box used in the study.

The students began the task by making predictions for dropping a marble each time (a total of 10), and then discussed the results. Initially, in Group1, Alicia thought that dropping the marble from one side or another might make the marble go to the opposite side at the bottom of the split-box, and she made her predictions based on that assumption. However, Emily did not agree with her and explained “because you see, it still goes straight down [showing the marble going through the funnel and coming down] and then if it hits there [the middle divider], it will go here [right] or there [left].” After the third experiment (see students’ predictions and results in Table 7), Alex said “I figured out that which ever one you put it again, it will go to opposite.” Although Alicia and Emily responded that it was “not always” because the marble might go straight and then go either way at the divider, this was the beginning of their investigation to look for a way to predict the results in the split-box.

Table 7. Predictions and results for 10-individual marble drops in Group1.

| Experiment # | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Side to drop the marble** | | L | L | L | R | R | L | R | L | L | L |
| **Results** | | **R** | **L** | **R** | **L** | **L** | **R** | **L** | **R** | **L** | **R** |
| **Predictions** | **Alex** | R | R | R | L | R | R | L | R | R | R |
| | **Emily** | L | R | R | L | R | L | L | R | R | R |
| | **Alicia** | R | L | R | L | R | L | L | R | R | R |

When making the predictions for the 4th experiment, Emily noticed a pattern in the results (R, L, R) and thus she predicted L "to see if the pattern will keep going." When the outcome was L, she was happy: "It's a pattern! It goes like R, L, R, L." In the 5th experiment, Emily predicted R and dropped the marble from R to investigate whether the marble would go to L or keep the pattern she discovered. After the result (L), she was convinced that "it sort of does matter which one you put it in." Hence, from that moment on, they all made predictions based on the "opposite sides" conjecture (see Table 7).

The students then made predictions about dropping all ten marbles together and conducted several experiments. Based on their investigations in the previous experiment, Alicia, Emily, and Alex conjectured that if the marbles were dropped from one slant (either the right or the left side of the funnel-like opening), more would go to the opposite side and a few of them to the other side at the bottom. For example, consider the following dialogue:

*Alicia*: I would say more to the left if you drop them from the right. If you drop them from the left side, more might go to the right.
*Sibel*: Okay. Alex, what do you think? [no response for a few secs.] If you drop all ten marbles, will we get more on one side or the other side, or will there be the same?
*Alex*: More on one side and less on the other.
*Sibel*: Why?
*Alex*: Because I think they are going to follow each other except for a few.

*Sibel*: Okay, so when you say more on the other side, how much more or less you guys think?
*Emily*: Maybe nine on one side and one on the other side. Or maybe eight on one side and two on the other side.
*Alicia*: Maybe if you drop more in over here [left slant],…seven through nine maybe [on the opposite side].
*Alex*: Maybe 5.

When Alex predicted five marbles on each side, both Emily and Alicia did not agree with him because they thought that in the previous experiments if you dropped them on one side, they mostly would go to the opposite side.

After this conversation, the students conducted various experiments with dropping all ten marbles as an attempt to inquire how much more marble would go to one side based on their current "opposite sides" conjecture. In doing so, they started with dropping six marbles from the left slant and four marbles from the right slant at the same time. Their initial predictions (before deciding the procedure of six from the left and four from the right) were 6R-4L (Alex), 8R-2L (Alicia), and 5R-5L (Emily). And the result came out to be 5R-5L. Although Alicia said she was surprised by the result (i.e., "I thought it would be like bigger than the five…maybe because there were bigger amount over here [left]"), Alex thought that it was "*pretty close* to 6-4 though." Next, the students wanted to examine 5 from L and 5 from R. Anticipation of a symmetrical dispersion of marbles began when students made their predictions as the following:

*Emily*: The same as last time [5R-5L] because there is even number on each side.
*Alicia*: Maybe more might land over here [L] this time because when we have more on here [L] and less over here [R], it was 5-5. Maybe 7L-3R.
*Alex*: More on R. 6 on the right.

While Emily stated another version of the earlier "opposite sides" conjecture, Alicia still expected unequal results based on the numbers they got in the previous experiment. In the first trial, Alex dropped one of the marbles from the left slant instead of right.

Therefore, they did another experiment, but again Alex did the same thing. Alicia and Emily were frustrated by Alex's disruption with the investigation at hand. Then, Alicia suggested that they drop all ten marbles lined up on the left slant. They all expected to get 2 marbles on the left and 8 marbles on the right and Alicia explained: "I think most of them would go in there [R]. But it depends on *how fast* they are going. Maybe like, if they are going *faster together*, then they might go different directions." It seemed that she was using a deterministic mindset based on the aspects of physics involved in the experiment, such as speed and moving together.

In the fifth teaching episode, the students continued their investigations with 50 and 100 marbles. Their initial predictions about 50 marbles were based on their previous conjecture at the end of the fourth teaching episode. For instance, Emily expected to get 1L-49R or 2L-48R marbles if they dropped all the marbles from the left slant. When I told them if they dropped all in the middle, their responses changed:

> *Alicia*: Maybe 25-25 or maybe 15-35.
> *Emily*: Half and half [25-25] or 15-35.
> *Alex*: 10-40.

The reason I asked this question was that the idea of dropping all the marbles in the middle came out of the students' investigations in Group 2 earlier. The student responses tended to be "even" or "close-to-even" except for Alex when they decided to drop the marbles in the middle. Once the result turned out to be 26L-24R, Emily and Alicia kept making predictions either "even" or "close-to-even" (i.e., 51-49, 101-99, around 250, and 499-501) for the larger number of marbles (i.e., 100, 200, 500, and 1000) with a fixed difference, such as 2, between the number of marbles on the left and those on the right side. For instance, some of the explanations they provided were: "[By Alicia for 499-

501] I thought that because this one was 26-24 and I thought it might be like one more in the other or two more in the other" and "[By Emily for 50-50] because of the results of 50 marbles, they might all *split equally* if they all go on that thing [the middle divider]." From these predictions, it could be argued that a difference of two (between the numbers of marbles in each slot) out of 1000 marbles is relatively smaller than the difference of two out of 50. Thus, the predictions of Emily and Alicia seemed to be consistent with the law of large numbers. However, Alex's responses (i.e., 80-10, 100-100 or 101-99) were inconsistent, with no solid explanation. Finally, when they did an experiment with 100 marbles, the result "47L-53R" was interpreted as "pretty close" by Emily and Alicia. Moreover, when I asked earlier, students did not interpret the results, such as 4-6 and 6-4, different than each other. Hence, it seemed that the result of 47-53 would not matter compared to their prediction of 51-49.

In Group 2, Caleb and Josh worked together on the Split-box task in the fourth teaching episode since Maya was absent that day. Therefore, I interviewed Maya at the beginning of the next session before all the students played the multi-level split-box game. When asked to predict and do the experiment for ten individual marble, Josh's question was where to drop the marble, from the left, or right, or in the middle. Then, I told him that it was up to them to decide what strategy they would follow in their investigations. Similar to the approach used by the students in Group1, Caleb and Josh decided to drop half of the marbles from the left slant and half of them from the right. Josh's explanation was as follows:

> *Josh*: That way we get a better sense of what happens like if one of them is dropped from here [right] and it goes over into this one [right], then if like 4 times it goes over to this [right] and one time it goes over into that one [left], then we know that when you drop it from here [right], most of the

> time it is going to come here [right]. Then we will have a sense of whether it will go over here [right] or over here [left]. If it went over there [left], we would know that every time you drop a marble, it is going to the same side or you might know that every time you drop a marble, or most of the time you drop a marble, it is going to go to different side.

Both Josh and Caleb made their predictions for each individual marble drop on the basis of "opposite sides" conjecture, like the students in Group 1. Their reasoning involved the aspects of physics relevant to the task, such as the path of the marble and the forces:

> *Josh*: I think that since it is already be going this way [towards left], it is going to be pushed that way.
> *Caleb*: When you roll it [from right to left], you expect it to keep going again this way [to left]. So, it can't like escape.

The results of four trials (L, R, L, R) convinced both students that the same reason would work for the next trials, and thus chose to stop their experiment with individual marbles at this point. Caleb's statement was quite clear: "There is no point of predicting anymore. It came true for the last four times."

Their investigation continued with dropping all ten marbles together from the right slant before I actually asked them to do the next experiment. Both were surprised when they watched that the marbles went to different sides. Their explanations were similar to the children's in Piaget and Inhelder's study (1975) when they talked about how the collisions of marbles during their fall dispersed them to one side or another.

> *Caleb*: There is too much that driven off the course.
> *Josh*: I am not sure why they all went on to different directions.
> *Caleb*: Maybe some driven off course.
> *Josh*: When one leaves the other, the other one goes to the opposite direction.

After this observation, their predictions for ten marbles dropped from the right slant all together were 6L-4R by Josh and 7L-3R by Caleb. They expected to have some marbles in the same side at the bottom due to the possible collisions while falling. Caleb also thought that it could be 5L-5R if they bumped into each other when they arrived at the middle divider and knocked one out to the other way.

Josh and Caleb began to understand the symmetrical dispersion of the marbles when they decided to let five marbles go from the right slant and five from the left slant. Specifically, Josh thought that this would make it a bit "more even" and they both predicted the result 5L-5R. However, Josh changed his prediction to 4L-6R and said, "to be on the safe side." This prediction was not exactly "even" but "close-to-even." Then, Caleb disagreed with him because he claimed "no number is being in a safe side in this game. [*Why?]* It could be anything, like 1-9." Josh believed that it could be anything if they dropped the marbles in the middle because the divider is right under the funnel. Based on these assumptions, Josh and Caleb did the experiment till they got 5L-5R when the previous results of two trials were 7L-3R and 7L-3R because Josh thought that they did not drop the marbles at the same time from each slant.

Their investigation into the symmetrical distribution of all marbles continued in the next trial where all ten marbles were dropped in the middle (the "middle" conjecture). Caleb explained his prediction, 5L-5R, by means of his expectation about the collisions of marbles and their symmetric dispersion: "It is going to be like, get knocked off, get knocked off,…[split evenly]." However, Josh noted that most of the marbles were "pushed off" to the left side in the funnel area at the top, and thus expected some deviation from an even split (his prediction was 3L-7R). When the result came out 7L-3R, Caleb said that Josh was right. Then, I wondered whether they thought 3-7 and 7-3 as the same result. Caleb did not think that it would matter because all possible results were equally likely to happen ("10% chance" for each, like 0-10, 10-0, 1-9, 9-1, etc.). However, Josh suggested that it would depend on where you were dropping the marble from: "If you were to drop them in the center, that wouldn't matter. [*Why?*] Most probably it would be 5-5 but you know it could be million of different things." When asked whether 0L-10R was as much likely to get as others, Josh thought that it was possible but "not too often" whereas Caleb believed "it is 10% [chance] for each."

Finally, in their investigation based on "the opposite sides" conjecture, Josh noted "strange" outcomes. For example, when they dropped all ten marbles from the right slant and the result was 4L-6R, Josh interpreted, "That was just *strange*. I mean you don't come across more *often*," based on the expectation that more marbles would go to the left compartment at the bottom. In addition, he was also surprised to get an outcome, like 5L-5R, when the marbles were let go from the left slant: "It's an even split. That wouldn't *normally* happen." It seemed that he would not have expected "even" or "close-to-even" results, believing that the symmetric dispersion usually occurs only when the marbles were dropped in the middle.

For the experiments with 50 marbles and 100 marbles, Josh made a generalization of the symmetric dispersion of the marbles from the previous investigations. As long as they were dropped evenly from each slant at the same time or all in the middle, he believed that the results would be "even" or "close-to-even." His responses, for example, included "somewhere near 25-25", "55-45 or near 50" because he thought that the marbles would be distributed symmetrically in the lower slots and the number of marbles would be "around the middle number." However, Caleb expected more deviation from "the middle number" when he made predictions, such as "35L-15R," "60L-40R," and "700L-300R." His responses were based on either deterministic and physical reasoning or the proportional model with no acknowledgment of the role of large number of marbles:

> *Caleb*: It is going to get knocked off again. Do we even room enough for? It's going to get knocked off again because I really think that they are going to be stacked up.
> *Sibel*: Why more left?
> *Caleb*: Until one gets the 40 and the other can't get blocked off and the rest of the 20 goes this way [left].
> …
> *Caleb*: 700-300 like it happens usually.
> *Sibel*: When usually?
> *Caleb*: You know 10 marbles go and 7 marbles on that side, except it's in hundreds. 700 on that side [left] and 300 in this side [right].

When I investigated students' understanding of the role of large numbers, like Piaget and Inhleder (1975), both Josh and Caleb expected more deviation from the middle as the number of marbles increased. Their explanations mainly were based on the previous results and how the marbles disperse during their fall:

> *Josh*: As we move up, the difference from the middle and the actual ones. As you drop more of them, the number from the middle, the middle number, the numbers that actually come out start to get further and further away from the middle number.
> *Caleb*: Larger! Because the more you have, the more it is going to go off.

As mentioned above, I interviewed Maya at the beginning of the fifth teaching session since she was not at school on the day of the fourth episode. Therefore, I present Maya's responses as a separate case in Group 2. For the individual marble drop experiment, Maya initially guessed that the marble was more likely to go to the right slot when she dropped the marble from the right slant. However, the marble went to the left side, and then Maya conjectured when she dropped it from the right, it went to the left, so if she dropped it from the left slant, it was going to go to the right. When the marble ended up in the left slot, she was confused and said "I think it [the box] is evil." Then for the following predictions, she usually guessed based on the previous results, i.e. "because it went to left more." When the result did not come out the way she predicted, she kept calling the box "evil" or "cursed."

When I asked Maya to make predictions about dropping all ten marbles together, she thought that dropping 5 marbles from the left slant and 5 from the right would not result in 5 in the right slot and 5 in the left slot. While dropping the first five marbles, she said "see they spread apart [3L-2R]." Then she was surprised when she got five marbles in each compartment because she did not know "it could happen." Unlike other students who worked on the experiments together, Maya did not attempt to conduct investigations in order to come up with a better way of predicting the results in the split-box. Instead, she relied on either the previous results and the proportional model, i.e., 30L-20R because "when I dropped 5 last time from here [right], 3 went to the right and 2 went to the left," or the physical bias in the mechanism, i.e., the right slant being shorter than the other one.

### Task 7: The Multi-Level Split-Box Game

As a subtask for the Split-box activity, I let students play the *Multi-level Split-box* game (Figure 23) which resembles the Galton box model of a binomial distribution. In this game, students moved the counters from the top of the sheet to one of the lanes (numbered 1 through 6) at the bottom, by dropping a marble in the split-box (five times for each counter) to decide whether to go right or left at each split in the multi-level diagram. They also marked the left and right turns on their counters after each marble dropping trial. At the end, students in each group had a discussion about the resulting distribution at the bottom slots. Before the discussion of results, I mentioned the term "distribution" again here. Then, I called the way the counters were arranged at the bottom lanes a "distribution of marbles." The reason I wanted to use this term again was to help them get used to it before the Hopping Rabbits task where they interpreted the distribution of random rabbit hops.

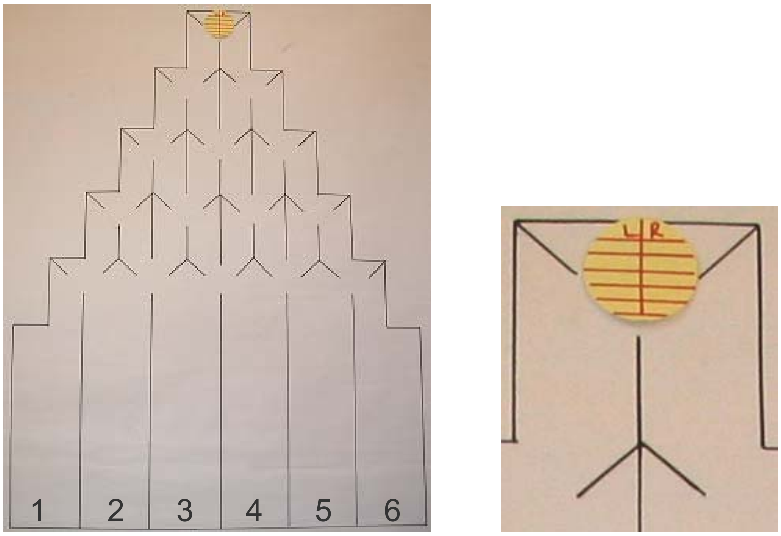



Figure 23. The Multi-Level Split-Box game with the counter [shown in the picture on the right] used in the study.

As in the previous tasks, students in Group 1 began the task by making predictions about where their counter might end up after five steps, by dropping the marble in the split-box each time. Students made their initial predictions based on either "just guess" (i.e., Emily's "3") or the expectation of getting the marble in the right slot more often (i.e., Alex's "4" and Alicia's "5"). Once they started playing the game, they all employed a strategy drawing upon the "opposite sides" conjecture that they investigated in the previous task with the split-box. More specifically, they often dropped the marble from the left slant to move the counter to the right in order to end up in the lane they predicted. For example, as can be seen in Table 8, Alicia expected to get to the lane 4 again in her second trial, and explained that "because I might do the same thing again if I keep putting it [the marble] here [on the left slant]." Then, as she said, she dropped the marble from the left slant each time, except when she realized that she needed to get L in her fifth marble-drop. Then, she put the marble on the right slant in the last trial, but it did not go to the left slot this time. The children's strategy did not always work, like in this case, and yet they believed in it. During the game, there was also no indication that the students expected symmetry in relation to the likelihood of getting the counters in certain lanes at the bottom, such as 3 and 4, 2 and 5, or 1 and 6.

Table 8. Group 1's results and predictions in the multi-level split-box game.

| | (Slant from which the marbles were dropped) **Results** | **Lane #** | **Predicted Lane #** |
|---|---|---|---|
| **Emily** | (LLRRR) **LRRLL** | **3** | 3 |
| | (RLRLR) **LLRRL** | **3** | 3 |
| | (LRRLR) **RLLRL** | **3** | |
| **Alicia** | (LLLLL) **RLRLR** | **4** | 5 |
| | (LLLLR) **LRRRR** | **5** | 4 |
| | (LLLLL) **RRRRR** | **6** | 4 or 6 |
| **Alex** | (LLLLL) **RLLLR** | **3** | 4 |
| | (LLRRL) **LRLLR** | **3** | 3 |
| | (LRRRR) **RRRLR** | **5** | 3 |
| **Sibel** | **RRRRL** | **5** | 4 |

After the game, I started asking the questions I prepared beforehand (see the task in Appendix D). When they were discussing that the counters mostly (four of them) arrived in the lane 3, Alex noticed that "they went to different patterns to get the same one." His clever observation raised a great opportunity to discuss different ways to get an outcome in terms of a varied sequence of L and R, which I did not plan in advance to have for this task. Looking at the results (the various paths on the counters) in the third lane, Emily and Alicia added that there were "so many ways" to get to the lane 3. Then, I asked Alicia what she thought about different ways to get to the lane 1 which had no counter at the end of their game. She tried different paths and concluded that there was one way to get to 1, "like one way to get there [6]." It seemed that she noticed the symmetry in the very end lanes. She also added "there is more than one way to get those [2 through 5]" but did not indicate any consideration for the symmetry in the middle lanes and the number of ways to get to these lanes.

As they started talking about the number of possible paths to get to each outcome, I followed up that thought by asking, "Are there less ways to get here [4] than here [3], or the same?" Emily's response was not convincing, i.e., "might be less, might be the same." She thought that it might take a long time to figure out "because you have to keep rolling the marble until we'll get a different pattern than the one already there." Although Emily had initially an understanding of possible permutations for two elements in the pre-interview task ("The Marbles"), she did not have a way to figure out those possible ways for five elements yet using the ideas of permutation and combination, nor did Alicia. She just looked at the ones they had when they played the game. In order to help them see what was common in the ones on the same lane, I asked them whether they saw any pattern in those paths. Alicia could only talk about the pattern of counters by whose counters were in the same lane, such as "Alex-Emily-Alex-Emily" in column 3 and "I am all alone" in column 6, and failed to recognize the same number of left-turns in a given lane.

After this game, I asked students to make another prediction if they were to play it 600 times. Alicia said that most of them would be in the middle lanes and there would be some in the lanes 1 and 2 if they kept playing the game. The reason she expected more in the middle was based on the previous results (i.e., she said, "in this experiment we had more in the middle [3, 4, 5]"). Similarly, the students thought that the fewest would be in the side lanes (1 and 6) because they believed that LLLLL and RRRRR would not often occur.

> *Alicia*: In this experiment, we had less in the sides too.
> *Emily*: It's *harder to get* like LLLLL and RRRRR than like LLRLR.
> *Sibel*: Why is it harder?
> *Alicia*: It keeps going to different directions [showing LRRRR path on the sheet as an example].

In their responses to the prediction question, I did not see any consideration of symmetry yet, except for the side lanes. Emily thought that they could get the same number of counters in some of the lanes, which could be any of them. Lastly, I asked them which lane they should choose before playing the game if they were to find a winner. Alicia and Emily said they would choose one of the middles, 3 or 4, because there were more of them in those lanes according to the results of their game. However, Alex wanted to make the game "more difficult" and "more fun" by choosing one of the side lanes (1 or 6), "the harder ones" he called.

Given that students in Group 2 often justified the distribution of marbles in the split-box based on either "the middle" conjecture or the "opposite sides" conjecture, they first started dropping the marble in the middle to move the counter each time, but eventually changed their strategy to dropping the marble from the opposite side in order to get to their predicted lane. As seen in Table 9, both Josh and Maya kept predicting that the counter might end up on the side lanes (1 or 6 in Figure 23) whereas Caleb predicted one of the middle lanes. When I asked them to explain how they made those predictions, they all provided some deterministic explanations by showing a path that might lead to the predicted lane:

> *Josh*: If you were to drop one, most likely it will keep on going down that way until it hits one of these walls [on the game sheet] and then bounce off that [right slant] and bounce off that [next right slant].
>
> *Caleb*: It starts from here [showing a path on the game sheet] It goes down here. Again. Again. Till it gets to this point [lane 5].
>
> *Maya*: It goes there [left], then it goes there, again it goes there, then it again goes there, and again it goes there [lane 1].

Table 9. Group 2's results and predictions in the multi-level split-box game.

| | **Results** | **Lane Number** | **Predicted Lane #** |
|---|---|---|---|
| **Josh** | LRRLR | **4** | 1 or 6 |
| | LLRLL | **2** | 1 |
| | LRRLL | **3** | 6 |
| **Maya** | RRLLL | **3** | 1 |
| | LLLLL | **1** | 1 |
| | LLLLL | **1** | 6 |
| **Caleb** | LRRRL | **4** | 5 |
| | RLRLR | **4** | 4 |
| | RRRRL | **5** | 5 |
| **Sibel** | LRLRL | **3** | 4 |

Their explanations did not imply any consideration of chance involved in dropping the marble in the split-box. Throughout the game, their initial deterministic predictions continued. Even though Maya noticed that the middle two lanes (3 and 4) were most likely ones "because so far they all went there [one on lane 4 and one on lane 3]," she later predicted "6" just because she wanted to get there, saying that 'we haven't covered it." Hence, the students' initial predictions suggested that they were not ready to talk about the most likely or least likely outcomes in this game yet due to the deterministic mindset.

In the discussions of the results, students were able to notice the most likely and the least likely lanes one could get to by looking at the results of their game. For example, Maya said that "most of them went to 3." Then, she started explaining how she got there by trying to remember each turn, but I suggested that could look at the path in her counter (RRLLL). In the following excerpt, she began to follow the path of right and left turns on the multi-level split-box sheet by looking at the marks on her counter:

> *Maya*: First, mine went right. So, I was right here [moving the counter on the sheet]. Then, I rolled it and it went right again. So, it went right here [moving the counter on the sheet].
>
> *Sibel*: And then what happened? You got left-left-left.
>
> *Maya*: Yeah, then I go to left [moving the counter on the sheet]. Left [moving the counter on the sheet]. Left [moving the counter on the sheet]. And went all the way to here [lane 3].

With her way of explaining the path to lane 3, I thought that this was a good opportunity to talk about different paths to the same lane. Then, I asked them a guided question to see how they might interpret results that were in the same place: "So, others ended up in that lane too. So, do you all get the same path to get there?" Josh responded that each counter in lane 3 had different paths (permutations), but there were three left-turns and two right-turns in all of them (combinations):

> *Josh*: These two, right here, are almost the same because on one of them it is RRLLL and on the other one it is LRRLL. So, both of them have three lefts and two rights.
> *Sibel*: So, how about the other one?
> *Josh*: This one is right here [LRLRL] is different.
> *Sibel*: How is it different?
> *Josh*: They would end up at the same spot. It's just that this one is more of a zigzag. Right here there is no zigzag.

There were also two counters in the lane 1 and therefore I asked them whether they thought those followed the same path as well. Josh said that "the ones on the corners are always going to have the same. [*Why?*] Because there is only one way to get there [showing LLLLL on the sheet]." Moreover, Maya recognized the symmetry in the multi-level split-box and added that there was only one way to get to lane 6 too. Then, I asked whether there was the same number of ways to get to 3 and 4. Only Josh noted the symmetry and said that there would be the same number of ways for each of the middle ones because he thought that the path for lane 4 would be just a reverse version of any path for lane 3. Others tended to show different paths on the sheet but could not reason about the number of ways. Eventually, Josh tried to generalize his notion of symmetry and the number of ways to get each outcome in this task by making a display "like a mountain" shown in Figure 24. He said that there were one way to get to the side lanes, 5 ways for each of the second and fifth lanes, and "most ways" for the middle ones, but he could not justify those numbers, expect for the side ones.

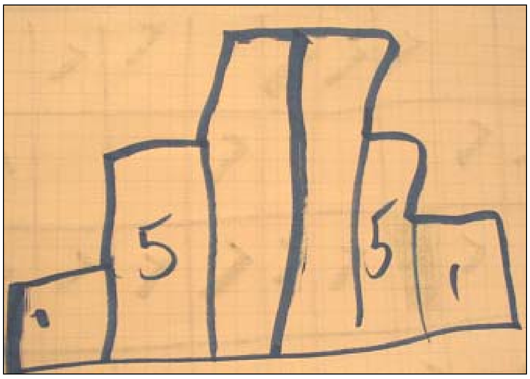



Figure 24. Josh’s drawing to show the symmetry and the number of ways to get each outcome in the multi-level split-box task.

When I asked them the prediction question for 600 marbles in the multi-level split-box task, their responses varied based on their own reasoning about the experiment. Consistent with his earlier conjecture about the number of ways, Josh expected to have most of them in the middle two lanes, then on the next two columns, and less on the sides. Further, he estimated those expected outcomes as follows: 225 in each of 3 and 4, “these have half of the chance”; 50 in each 2 and 5, “those don’t have even half of the chance”; and 25 on each of the side ones, “hardest to get into.” When Caleb predicted that most would be on 4 and 5, then on 3 and 2, and then 1 and 6, he thought that the marble had a tendency to go to the right “when you least expected” and hence it would always go in one of those two (4 and 5). Maya’s predictions were basically based on the results they got by playing the game ten times: “most on 3,” “second most on 1 and 4 and 5,” “less on 2,” and “less less less on 6.” Finally, when I asked them which lane they should choose before playing the game if they were to find a winner, all of them responded based on their reasoning about the most likely outcome for 600 marbles.

Overall, as I conjectured initially based on Ughi & Jassó (2005), the students tended to recognize and anticipate the symmetry (limited to the side lanes), except Josh, who indicated the symmetry around the center. Moreover, they all expected more counters in the center mostly based on the previous results and few on the sides because they thought LLLLL and RRRRR were hard to get. Even though students showed different possible paths to go to a specific lane on the multi-level split-box representation, only one student, Josh, noted the same number of left-turns in a given lane when discussing the different paths to get to the same place. Hence it could be argued that students might be ready to develop a way to find out the permutations (all possible ways) of left and right in five turns, but might not necessarily think of different combinations (i.e., 2 left and 3 right turns, 1 left and 4 right turns, no left and 5 right turns, and so on) in relation to all possible ways to get them.

## Task 8: Bears Task

From the discussions in the previous task where students talked about the possible ways to get to a particular outcome, such as the bottom lanes in the multi-level split-box, based on the results in their games, I found that students did not currently have a way to generate all those possible ways. Since I initially conjectured that an understanding of permutations and combinations was necessary for the discussions of the random rabbit hops in the Task 11, I gave the *Bears Task* to the students to examine whether they could together generate all the different ways to arrange five bears in a row, using the red and blue bears.

In both groups, students began to arrange the blue and red bears in a row on the floor (see Figure 25) and each time they made a record of that arrangement with tally marks on a piece of paper using the red and blue markers (see Figure 26). Also, all of them easily recognized the symmetry when generating reversed color sequences of five bears. For example, when the students in Group 1 had RBBRR[1], they easily came up with BRRBB next.

Figure 25. The Bears Task.

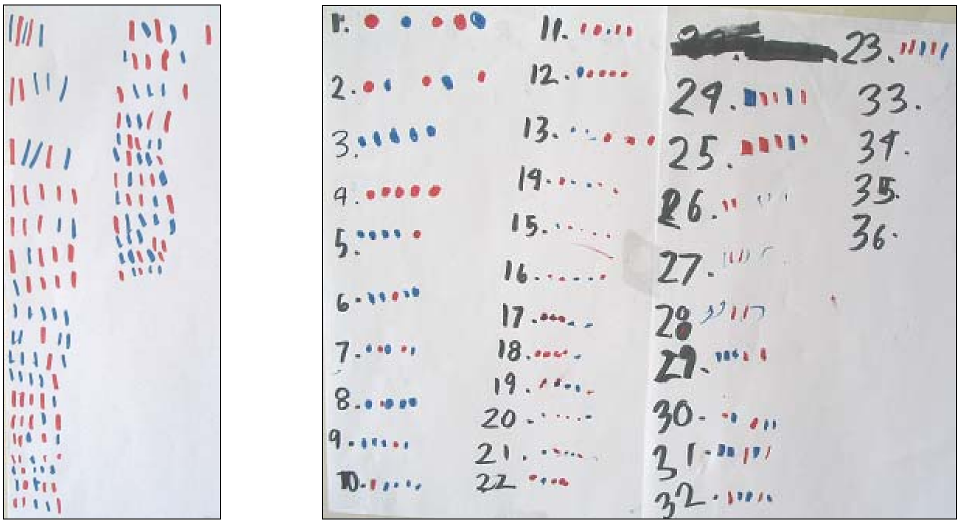


Figure 26. Students' inscriptions for finding all possible ways to arrange blue and red bears in Task 8 (Picture on the left by Group 1 and picture on the right by Group2).

Using the "opposites" to find another possible arrangement of five red/blue bears was the only strategy employed by Group 1. However, since they did not do this

[1] In this section (The Bears Task), R denotes a red bear and B denotes a blue bear.

systematically, they could not keep track of the "opposites" in their list. They mainly relied on a trial-and-error method. Also, they recognized the importance of order in the task, particularly when Alex changed the place of the blue bear and made a different arrangement, such as BRRRR and RBRRR. Eventually, they were able to figure out thirty of all possible thirty-two permutations (Figure 26).

The students in Group 2 also started with guessing different arrangements by trying. When Caleb found RRRRR and BBBBB, he called them "the simplest ones" because they all were the same color. They also used the "opposites" to find different possible arrangements. During the task, Josh came up with a strategy to arrange blue and red bears for each combination when he noted the pattern of 1R4B, 2R3B, 3R2B, and so on. His strategy involved coming up with an ordered list of those different combinations, such as RBBBB, BRBBB, BBRBB, and so on. In doing so, he just moved the red bear to the right each time. When listing the permutations for 2R3B, he started with RRBBB and then moved the two red bears to the right one at a time. Next, two red bears were moved to the right one apart from each other, such as RBRBB, BRBRB, and BBRBR. And then, he continued to take them apart by two, i.e., RBBRB, and by three, i.e., RBBBR. Caleb followed Josh's strategy and helped him generate all permutations, but Maya chose to work on her own to find the possible ways just by trial-and-error and the "opposites" strategy, like the students in Group 1. While Josh and Caleb were sure that they got all possible ways to arrange blue and red bears once they got thirty-two of them with the strategy they used (Figure 26), Maya was not sure whether there were still more ways to do and indicated that she would try more until she found all of them.

It was evident that when Josh discovered the different combinations of red and blue bears, he was able to come up with a strategy to find all permutations for each combination. Hence, this example seemed to support Piaget and Inhelder's (1975) claim about the discovery of permutations following the development of the idea of combinations. According to Piaget and Inhelder, "…combinations consist simply in associations effected according to all the possibilities, while permutations, which are much more numerous, imply an ability to relate according to a mobile system of reference (transformation of the starting order for variable initial elements)" (p. 194). Hence, consistent with their argument, Josh's strategy involved starting with an order, like RRBBB, and then he transformed it to generate the successive permutations by changing the order in a systematic way.

### Task 9: Coin Flipping Activity

Prior to the *Hopping Rabbits* task in which students were asked to simulate the rabbit hops by flipping a coin, I wanted to examine their conceptions and reasoning about coin flips both in individual and repeated trials. At the beginning of the task, students discussed the purposes of flipping a coin and the possible outcomes based on their personal experiences. They were all familiar with the use of coin flipping to make decisions or choices in daily life and in sports. Also, a common perception among the students was one's inability to predict the outcome. Alicia and Emily thought that you would need a "psychic power" to predict the outcome correctly. Some of the responses regarding the uncertainty of coin flips were as follows:

> *Alicia*: You can predict which way marble will go if you drop it from one side, but not the coin toss.

…
*Emily*: You never know.
…
*Maya*: It could be anything!

When students in Group 1 started to make predictions for five individual coin-flips (see Table 10), their initial belief about unpredictability (or uncertainty) of the coin flipping result were persistent through out this activity. Therefore, they said that their predictions were "just guess." When the result was the same as their predictions, Alicia and Emily said that "we are psychics."

Table 10. Group 1 student predictions for 5 individual coin-tosses and results.

| **Trials** | **Alicia** | **Emily** | **Alex** | **Results** |
|---|---|---|---|---|
| 1st | H | H | H | **H** |
| 2nd | T | T | T | **T** |
| 3rd | T | T | H | **T** |
| 4th | H | H | H | **T** |
| 5th | H | H | T | **H** |

Then, the students were asked to make predictions about many trials, such as 10, 50, 100, 200, and 1000 coin-flips. While Alex's predictions depicted an inconsistent way of reasoning, Alicia and Emily, who made the predictions together, said that the result might either be "even" (i.e., 5T-5H, 25T-25H, 50T-50H, 100T-100H, and 500T-500H) or have "at most 10 difference" (i.e. 40T-60H, 110T-90H, 510T-490H). The difference of 10 seemed to be a reasonable number when the results did not come out to be even. Since the difference of 10 was expected for all different number of trials, they did not consider the role of large numbers. Similar to their thinking for the marble dropping in the split-box, they seemed to believe that 40T-60H was the same as 60T-40H. Therefore, they did not think about making clear which of them (Heads or Tails) might be 10 more in a given number of trials.

Students in Group 2 also experimented with the individual coin-flips in five trials and made predictions before each trial (see Table 11). For the 1st trial, students' predictions were mainly based on their personal experiences:

> *Maya*: I always guess Tails.
> …
> *Josh*: Tails never fails and it depends on how you are doing it.
> …
> *Caleb*: I normally say Heads.

Table 11. Group 2 student predictions for 5 individual coin-tosses and results.

| **Trials** | **Maya** | **Josh** | **Caleb** | **Results** |
|---|---|---|---|---|
| 1st | T | T | H | **T** |
| 2nd | T | H | H | **T** |
| 3rd | H | T | H | **H** |
| 4th | H | H | H | **H** |
| 5th | T | T | H | **T** |

For the 2nd trial, Josh conjectured that "once Tails never fails, Heads never fails" and used the same reasoning for the rest. While Maya just said that she likes Tails, Caleb kept predicting Heads throughout this activity. In the next one, Maya wanted to switch to Heads since she believed that the outcome probably would alternate, like THT and so on. Once the results happened to be TTHH, she said "I am going to do a pattern [TTHHT]." Unlike the students in Group 1, these students tended to believe there was a pattern in the coin flips.

Students' predictions for many coin-flips, such as 10, 50, 100, 200, and 1000 revealed quite similar reasoning as in the marble dropping activity in the split-box. For example, Caleb again thought that "everything is possible" indicating that all outcomes, like 1T-9H, 2T-8H, 3T-7H etc., were equally likely, but he chose the one he wanted it to be, like 5T-5H. Furthermore, there was not a really consistent pattern in his predictions, such as "26T-24H; 60T-40H; 100T-100H; 500T-500H." Like in the previous task, Josh used his idea of "split in the middle" and "around the middle" for every prediction (i.e., "26T-24H; 55T-45H; and 100T-100H"). For instance, he said that "It's kind of the marble game. The middle is always going to be the one most likely. Most likely it's going to *split in the middle*." Again, Maya usually employed the proportional model to make her predictions, i.e., [Her prediction was 700T-300H] "Because of 3T-7H" (in the previous prediction for 10 tosses). Josh also seemed to use the proportionality when he argued that the difference from the middle would get larger as the number of coin-flips increased. For instance, he explained his prediction of "600T-400H" by saying, "For 1000, most likely it is going to be bigger difference. Like with 10 marbles, 6 to 4, and with 1000 marbles, 600 to 400. The difference keeps getting bigger." Hence, Josh was thinking of absolute differences when he suggested that the difference between the numbers of Tails and Heads would get larger proportionally (i.e., 2 out of 10 vs. 200 out of 1000) as the number of tosses increased. According to Piaget and Inhelder (1975), the thought of absolute differences, rather than the relative ones, would indicate no consideration of the law of large numbers.

I found some similarities in students' thinking between the (simple) split-box task and the coin-flipping task, since they both were models for equiprobable outcomes, except that students interpreted the predictability of the outcome differently in each task. When making predictions before each coin flip, students' responses suggested that there might be some pattern-based reasoning in relation to their conception of randomness in flipping a coin, such as Tails after Heads, or a pattern like TTHH. In their predictions for the repeated trials, students tended to predict "close-to-equal" or "even" outcomes (the number of Heads and Tails). Also, students did not seem to be ready to talk about the role of large number of trials when making their predictions, particularly based on the proportional model or “10-difference.”

### Task 10: Spinner Task

In this task, students made predictions for the outcomes of a spinner with three equal-sized parts (yellow (Y)-blue (B)-purple (P)) with respect to different numbers of spins, such as 5, 10, 20, 30, 100, 300, 1500, and 2000 spins. Since the spinner involved three equiprobable outcomes, the most common way students (except Alex) used to make predictions for 30, 300, and 1500 spins was to guess even number for each color, such as 10Y-10B-10P, 100Y-100B-100P, and 500Y-500B-500P, no matter what the number of trials was. It was also consistent with earlier responses that Alicia, Emily, and Josh provided in the previous tasks, such as the split-box and coin flipping. When the result could not be evenly split, their predictions were still “close-to-even,” i.e., Alicia and Emily’s “2Y-2B-1P; 4Y-3B-3P; 7Y-7B-6P” and Josh’s “2Y-2B-1P; 4Y-3B-3P; 7Y-7B-6P.” Again, based on his previous reasoning “around the middle split”, Josh made his predictions for 100 spins as “33Y-32B-35P.” However, for 1000 spins, he relied on the proportional model, i.e., “700Y-700B-600P,” because he always expected larger difference when the number of trials increased.

When the students discussed their predictions in each group, I found that some students tended to predict extreme ones, such as “[Maya] 0Y-1000B-1000P,” “[Caleb] 1000Y-1000B-0P,” and “[Alicia and Emily] 1800Y-200B-0P.” I called them “extreme” because they predicted none on one color when the spin number was as large as 2000 even though they made “even” or “close-to-even” predictions for the previous ones, especially for 300 and 1500 spins which were "even split." In each group, those predictions were discussed among the students because neither Josh nor Alex agreed with no spin on one color. For example, Alex thought that Alicia and Emily's prediction "1800Y-200B-0P" was unusual because he said "you are bound to have purple if you spin it 2000 times." Also, Caleb wanted to change his prediction "1000Y-1000B-0P" after Josh's comment, "it is going to land on every color." Then I asked Caleb whether it would be possible to have 0 spins on purple, like his first prediction "2Y-3B-0P." He did not think so because he believed that it was a small version of "1000Y-1000B-0P." The proportional model again existed in his thinking for this particular example. In terms of the role of large numbers, it seemed that students did not have an understanding of a smaller variability in the number of spins on each color from a large number of trials. Although students did not do the experiments for their predictions, some of them (Caleb and Alicia) pointed out that it would depend on "how hard/well/fast" you spin, when asked to explain why they made their predictions. Hence, they also had some causal and deterministic reasoning about the possible outcomes of a spinner experiment if they were to spin it.

### Task 11: Hopping Rabbits

Prior to the sixth teaching episode, the students engaged in concrete experiments that generated various types of distributions, such as a centered distribution with the chips and a uniform distribution with the marbles and the split-box. Even though students began to examine a binomial distribution in the multi-level split-box game, it was not until the *Hopping Rabbits* activity that they got involve in deeper discussions about how the distribution was generated, what were the possible outcomes (the sample space), how the results of the simulation (both with the coin simulation and the computer simulation) turned out to be on the possible outcomes, and how the results could be used to reason about the probabilities of each possible outcome in the sample space. This task was completed over the course of approximately five teaching episodes. As a result of the retrospective analyses of these five teaching episodes, this section has been organized into seven subsections to present the findings from both groups: (1) the initial predictions; (2) the possible outcomes (sample space); (3) the use of inscription of paths; (4) the privileged side vs. the symmetry in the distribution; (5) the most likely and the least likely outcomes; (6) the number of ways that an outcome can occur; and (7) the quantification of the probabilities associated with the sample space.

**(1) Initial Predictions:** The task started with the presentation of an uncertain situation which was the following[2]:

> Suppose there are a number of rabbits on a land and each rabbit can choose to hop only right or left. For each hop, rabbits are just as likely to hop right as left. We want to know where a rabbit is likely to be after 5 hops.

The students in Group 1 began to talk about some possible paths for five rabbit-hops (see Figure 27) given the situation stated in the problem. Moreover, Emily argued that "You never know. It could be LRLRL, LLRLR, RLLRR…" Both their verbal responses and their inscriptions (shown in the Figure 27) suggested that the rabbit hops were interpreted as uncertainty, rather than a deterministic situation. Recall that these students often reasoned with deterministic and causal explanations when they interpreted

[2] An adaptation of the Binomial Rabbits model in Wilenksy (1998).

the distributions of the buffalos, the leaves under the tree, and the airline routes in the first task.

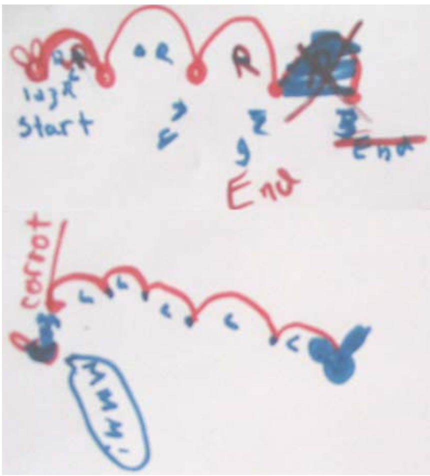



Figure 27. Possible paths for five random rabbit-hops (Group 1).

In Group 2, when Maya thought that the rabbit could go like a zigzag and forward, Josh wanted to explain to her that the rabbits in this problem could only hop right or left. Then, as a way to show what he thought about the rabbit hops, he generated the top inscription in Figure 24. In this inscription, the starting point (the boldest middle dot) and five other possible places to hop on each side of the starting point (left and right hops) were marked. Once they started making different paths in terms of left and right hops, like the students in Group 1, Josh and Caleb realized that some of these places on the right and left side of the starting point were impossible to get after five hops. Then, those places were crossed out on the bottom inscription in Figure 28. This observation was relevant to an important property of the binomial rabbit hops as being a discrete distribution, rather than a continuous distribution, i.e., a distribution of the sum of two dice.

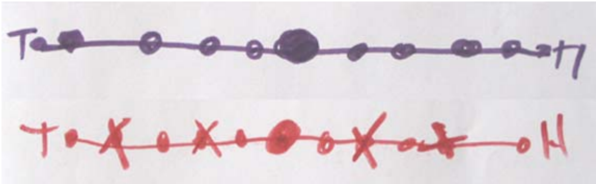


Figure 28. Possible and impossible outcomes for five rabbit-hops (Group 2).

Before introducing the idea of flipping a coin to decide where the rabbit might hop each time, at the moment I decided to ask the students in Group 2, “How can we decide which way the rabbits might hop?” Maya’s response was an interesting one as she suggested that we could look at the rabbit’s stomach as a sign. In her explanation, she indicated that her father told her in playing basketball “if you want to know which way somebody is going, look at their stomach because their stomach will move to the way they are going to go.” Her consideration of a *sign* to determine the possibility of a future event was similar to the concept of probability in medieval ages (the concept of sign as evidence in *opinion*) that was presented in the history section of Chapter 2.

When I suggested that students could flip a coin to decide which way the rabbit might hop on a number line representation, which was similar to Josh’s inscription earlier (Figure 28), they instantly made a decision about which side of the number line would be represented by Tails and Heads. Students’ first predictions suggested that majority of the students in both groups just guessed, rather than considered the possible outcomes and paths, when thinking about where the rabbits would most likely be. For example, Alicia said that the rabbits might end up on the right after five hops by “just guessing.” Then, I asked her to specify which places on the right side and she replied, “around 4, 5, and 6.” Emily preferred the area around 3, 4, and 5 since she thought that “if it keeps going right, it could be 6 because we are flipping it five times. And if it goes left, it can’t be 6.” Even though she mentioned that they would be flipping the coin five times, she did not realize the possible range within the number of hops yet. Given that Josh and Caleb already started thinking about various paths and the possible outcomes for five hops, both made their predictions based on some possible sequences of Heads and Tails. For instance, Josh thought that -/+ 1s would be the most likely ones because there was a 50-50 chance of getting Heads or Tails and therefore the rabbits would hop back and forth mostly. Moreover, Caleb predicted -/+ 3s, indicating that they both had the same chance because they were just the opposites in terms of the paths, such as HHHHT and TTTTH. Even though Caleb did not justify the choice of -/+3s, both he and Josh seemed to notice the symmetry around the 0.

**(2) Possible Outcomes (Sample Space):** Right after their initial predictions (in Group 2), Josh pointed out the impossible outcomes on the number line: "Don't say anything above 5, or 4 or 2 because they are impossible." However, 0 was not noticed as an impossible outcome until they got their coin simulation results. Unlike this group, students in Group 1 failed to notice any of the impossible outcomes prior to the discussion of the simulation results shown on the stacked plot (see Figure 29). For example, when Alicia noted that there was none beyond -/+ 5, Emily explained that "because they can't go further than 5 because you flip the coin only 5 times." Flipping the coin five times was now evident to Emily when reasoning about the range of possible outcomes in this particular situation. Moreover, during the discussion of results, Alex noted a pattern above the number line, saying that "X-nothing-X-nothing-X-nothing-X-nothing…" Nonetheless, no one could justify why the results occurred in that pattern at this moment. However, earlier Emily noticed that there was no rabbit on 0 (the starting point) and when I asked them why, they started making different paths to try to land on 0. After several trials, they seemed to be convinced that 0 was not possible after five hops. At the end of the discussion of the simulation results, I asked them a guided question about whether there was a possibility of landing on -/+ 2 or -/+ 4. Then, students began to think about possible paths which might end up on the even numbers. When they failed to find a path to land on 2 after four or five trials, they seemed to be convinced that the rabbits could not get to the even numbers by hopping five times (i.e., “Emily: If you go like 1, 2, 3, 4, and you still have a hop left.”). Finally, Emily made a conjecture that if there were an odd number of hops, then they would land on odd numbers. The same conjecture was made by Caleb in Group 2 also, right after their initial predictions prior to the coin simulation. Furthermore, this conjecture developed by students in both groups was generalized to the other events when they started to run computer simulations for 10 hops (an even number), instead of 5 hops. Hence, they all expected the possible outcomes to be even numbers on the number line between -10 and 10 for 10 rabbit-hops.

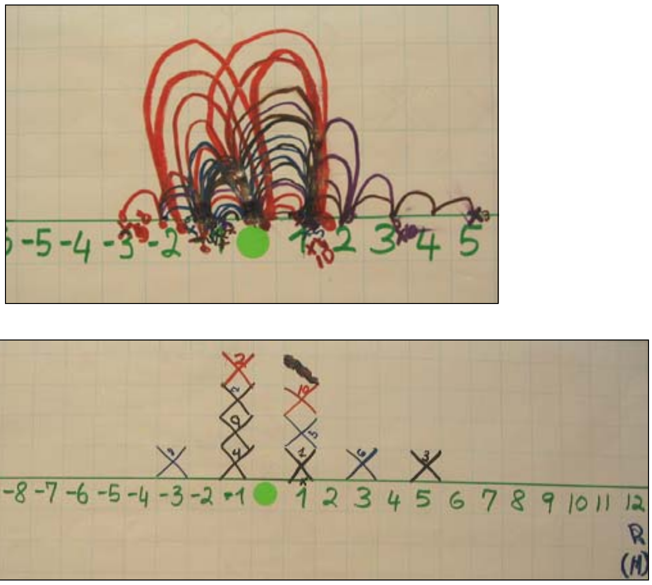



Figure 29. Students’ inscriptions for the simulation of the rabbit-hops in Group 1.

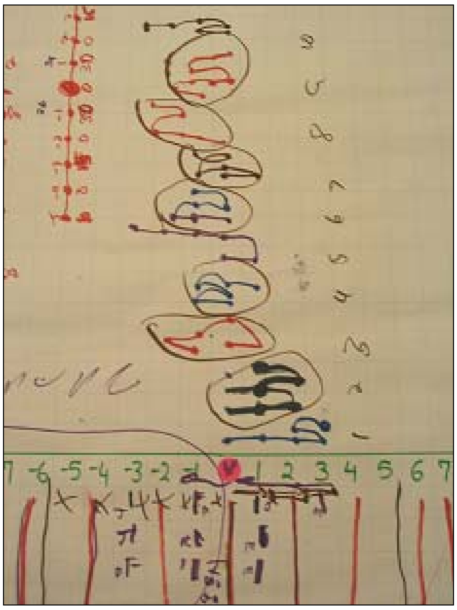


Figure 30. Students' inscriptions for the simulation of the rabbit-hops in Group 2.

**(3) Use of Inscription of Paths:** Like in the initial predictions about possible hops (Figure 27), students recorded their simulation results using "paths", i.e. arches (see Figure 29) and connected dots (see Figure 30). I called those artifacts, "student-generated inscriptions," drawing upon the work of Lehrer and Schauble (2000) which focused on the evolution of student-generated inscriptions and their mathematical arguments as students described and reasoned with distributions, by incorporating Latour's (1990) ideas in understanding the practices of scientists through the evolution of inscriptions. Latour used the term 'inscriptions' to refer to a broad range of symbolic tools including diagrams, lists, drawings, equations, writing, maps, and so forth. More specifically, Latour (1990; 1999) noted that inscriptions were tools for representing aspects of the world, and communicating and persuading one's ideas to the public, as well as transforming them into new forms of ideas. Hence, they were central to constructing arguments within scientific communities.

Similarly, in this study, the students transformed the binomial rabbit hops into paths and stacked plots of final positions on a number line (Figures 27, 29 and 30). For instance, students were initially asked to predict where the rabbits were more likely to be after five hops before the coin simulation. Then, Alicia and Emily began to make a drawing of some possible hops of five (Figure 27). Alicia made a particularly detailed inscription of a possible path of five hops by labeling each hop as right or left, first, second etc., and labeling the start and end points. Another example of inscription was generated by Josh to show possible and impossible outcomes by marking the final positions after five hops on a straight line (Figure 28). Unlike Alicia and Emily's, the inscription generated by Josh seemed to capitalize on the possible final outcomes rather than the possible paths leading to a particular final position on the line. When asked to simulate rabbit hops by flipping a coin five times and to make a record of their final positions, students in each group tended to show the paths of five hops for each trial, either by drawing arches or connected dots (see Figures 29 and 30), so that they could have a record of each hop before the rabbits ended up in a final position. Students also were prompted to present their data in a way that permitted comparison of the likelihoods of each outcome based on the frequencies. Then, the students in Group 1 created a separate display of rabbits after five hops by using the stacked dot plot that they knew how to create from their classroom experiences. In Group 2, students chose to show the frequency of rabbits right beneath the number line, above which they made a record of each path.

The student-generated inscriptions were also relevant to a particular purpose. They often used them for supporting their verbal arguments, in particular for justifying the impossible outcomes on the number line and the different number of ways to get to the same outcome. For example, in Figure 31, the students in Group 2 began to make their predictions about the most likely places for rabbits to land on after 10 hops. In their discussion of the most likely outcomes, the emerging idea was the number of ways to get an outcome. When students were prompted to think about for which outcome(s) there were more numbers of possible ways after 10 hops, Caleb responded that there were more ways to land on 2. Then, in order to make his argument, he came up to the board and began to draw the possible paths to get to his predicted most likely outcome. Below is the excerpt from Caleb's way of displaying different ways to get to 2 on the number line:

> *Caleb*: Here is why. Heads, Heads, Tails, Tails, Tails, Tails [paused to count the number of hops so far]
>
> *Sibel*: You have six.
>
> *Caleb*: Heads, Heads, Heads, and Heads. And that's on 2. And you could have done the opposite with Tails. And you could have done Heads, Heads, Heads, Heads [paused to count the number of hops so far], Heads [pause], Heads, Tails, Tails, Tails, Tails. I can't think of any more right now but.

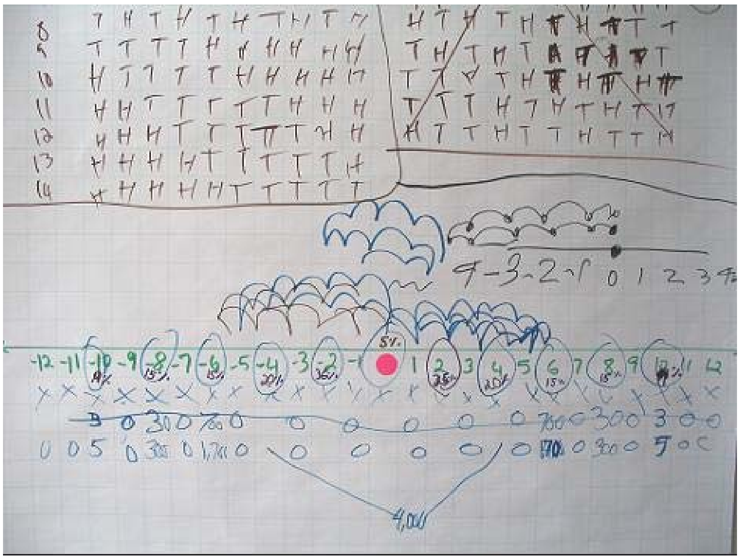


Figure 31. Student-generated inscriptions for figuring out the number of different ways for each outcome after 10 hops.

Latour (1990) also mentioned that inscriptions were not simply copies of the world, but adaptable representational structures to serve a particular purpose and thus have their own "affordances and constraints" (Wertsch, 1998) in supporting argumentation. For example, in Group 1's inscriptions with the arches (at the top in Figure 29), students were able to keep track of five hops each time they simulated five rabbit-hops. Although where the rabbits landed most was visible by looking at the arches, i.e. "Alicia: More lines in the middle…less line on the sides," this inscription did not easily support reasoning about frequency of rabbits on possible outcomes on the number line, unlike the stacked plot, which however did not have the track of each hop (at the bottom in Figure 29). When one or more inscriptions developed as a way to represent the various characteristics of a phenomenon, Latour referred to those as the "cascade of inscriptions." For example, the stacked dot plot in Figure 29 was developed from the earlier inscription with the arches representing the paths of five hops for ten trials.

Further, the student-generated inscriptions in this task had certain properties, which were mentioned in Latour (1990). For instance, these inscriptions were (1) *mobile* since they were recorded on the sheet; (2) *immutable* since they would not change their properties when transported; (3) *flat* because they were two-dimensional materials; (4) *scalable* as they were easily rescaled to include them in this paper with no change in their internal relations; and (5) easily *combinable* and *superimposable*. For instance, the students in Group 2 layered inscriptions by combining "paths" with simulation data in Figure 30, which made each path of five hops as well as the frequency of rabbits on each outcome visible.

**(4) Privileged side vs. symmetry in the distribution:** Even though each of the paths simulated by a coin flip (i.e., HHTTH, HTHTH, HHHHH, TTTHH, and so on) are equally likely to happen, the final positions after five hops produces a different pattern with a symmetrical distribution, i.e., any combination of 3 Heads and 2 Tails (and 3 Tails and 2 Heads) is the most likely outcome, any combination of 4 Heads and 1 Tails (and 1 Tails and 4 Heads) is a less likely outcome, and any combination of 5 Heads (and 5 Tails) is the least likely outcome. In this study, the students mostly were able to see the symmetry in the distribution of rabbits after five hops. And usually the symmetry was associated with expecting "even" on both sides. For example, Josh always made his predictions for "more on 1s" (both positive and negative ones), "then on 3s," and "few on 5s" (i.e., "6 on 1s, 3 on 3s, and 1 on 5s") and said that it could be even in both sides (5 and 5 on positive and negative sides). Caleb also made alike predictions, such as equal on both sides and overall number of rabbits on each symmetric outcome: "70 on 1s, 20 on 3s, and 10 on 5s."

Consistent with their previous conception in the earlier tasks (the *Split-box* task and the *Coin Flipping* task), Emily and Alicia often predicted "even" or "around even" number of rabbits on each side, e.g., Emily: "I think it is most likely to be equal on both sides no matter where they are" and Alicia: "Probably not. You don't really get even every time." They also made their predictions somewhat more specific, such as "more on 1s than 3s and than 5s," as they conducted more and more simulations using the computer simulation. Alex seemed to expect "more rabbits on 1s" and "1s more than 3s", but he never justified those predictions with any explanation. Therefore, it made it hard to speculate about his reasoning in relation to the symmetry.

Throughout the task, however, one student expected one side to be privileged. For instance, Maya consistently predicted more rabbits on the left (negative) side because there were more on the negative side when they did the simulation of rabbit hops with the coin. She also did not want to make a specific prediction for each outcome because she thought that anything could happen. Even though they did more and more trials with larger number of rabbits, such as 100, 500, 1000, 10000 rabbits) with the computer simulation, she did not tend to change her prediction about "more on the left."

In the last episode of the *Hopping Rabbits* task, students were asked to make predictions and run simulations when the chance of hopping right was modified to "75%" and "25%" by using the feature of the NetLogo computer environment in order to investigate their perceptions of symmetry when the chance of hopping right was changed. For a 75% chance of hopping right, students' initial predictions for the most likely places for landing after 10 hops were quite reasonable. For instance, Emily, Alicia, and Alex predicted that there would be more rabbits on the right side, such as on "2, 4, and 6 area," because there was more chance of hopping right. Similarly, others also expected to get most of the rabbits on the right side. For instance, Caleb thought that the rabbits were most likely to be on 2 because the "middle" would shift to the "2 area" with the 75% of chance of hopping right, so did Maya. However, Josh did not agree with their prediction, by expecting more *skewness* in the shape of the distribution and said that "since we have more chance of hopping right, all of the numbers is going to be *pushed over*. So, it's not going to be 2 because it's too close to the 0. It could be 6, 8, or 4. I'd probably guess 8 or 6." After the simulation, Maya tended to make a conjecture about predicting the most likely place for landing after 10 hops for different chances of hopping right. For example, she suggested that the numbers in the number line corresponded to the chances of hopping right, such as 6 to 75% and 8 or 10 to 90% chance of hopping right. After these discussions, all students made their predictions for 25% of hopping right based on the symmetry in the distribution with respect to the middle (0), i.e., "Josh: -6 because 25% to the right is the same as 75% to the left."

**(5) Most Likely and Least Likely Outcomes:** After students simulated the five rabbit-hops by flipping a coin, and began to discuss the data generated, I asked them why there were more rabbits on -/+ 1 more and few on -/+ 5. Like the participants in the pilot study, they tended to identify -/+ 1 as "easier" and -/+ 5 as "harder to get." For example, Josh responded that "50-50 chance that the coin will land on Heads or Tails. So, most likely it is going to go something, I mean you can see a lot of zigzags. [He is circling all the paths for 1s in their representation] You can see bunch of zigzags." With "zigzags" Josh meant to refer to "back and forth" hops, like THTHT or HTHTH, but he circled all the paths for -/+ 1, including HTTHT and HTTTH on their inscription (Figure 26). Hence, the examples of "easy to get," which students provided to justify why -/+ 1 was the most likely outcome, varied, such as Caleb's "THTHT" and "HTHTH," Emily's "HHHTT" and Alicia's "HHTTH," but the number of Heads and Tails in each path was common in all. Then, their responses about the likelihood of -/+ 5 were mostly based on their belief about getting the same side of the coin in 5 tosses. Josh, for example, said that if it went like HHHHH or TTTTT, it would mean that "you are pretty lucky because that doesn't happen very often." Moreover, Alicia reasoned that "It's harder to get there (-/+ 5) because they usually go like [RRLLR] and less like [RRRRR]." Alicia and Emily also thought that "it would be a miracle to get all on 5s." Accordingly, the students believed that it was rare to get 5 Heads or 5 Tails. Given their explanations for both the most likely and the least likely outcomes, it could be argued that the reason they expected the five coin-tosses to be more like THTHT or HTHTH or HHHTT or HHTTH, and less like HHHHH and TTTTT, was possibly due to an expectation to get "even" or "close-to-even" numbers of Heads and Tails in five coin-tosses, similar to their reasoning in the *Split-box* task and the *Coin Flipping* task. However, a normative reasoning would involve an expectation of having more ways (permutations) to get 3 Heads and 2 Tails (combinations) than to get 5 Heads or 5 Tails (shown in Table 4 earlier in this chapter) to explain that the likelihood of outcome 1 was more likely than that of outcome 5.

When students were asked to make predictions for the most likely place to land on after 10 hops, most students seemed to expect some deviation from the middle (0) as the number of hops increased. Some examples of those responses were as follows: "Emily: Somewhere around 4, 6, and 8 area and more on 4 and 6 because they are the middle area between 0 and 10" and "Alex: On 2 because it's twice as much as 5." Moreover, Josh thought that the rabbits would spread out more on 4s than on 2s and 0. Once they started running computer simulations for 10 hops with 10000 rabbits, they revised their initial predictions: Emily: "Near 0 most and less on 10s" and Alicia: "Middle, more on 0 than 2." Then, similar to the reasoning they used for five hops, Emily expected more on 0 because "the bunny rabbits mostly go like, RRLRLLR… and not like, RRRRLLRR…" It seemed that she anticipated more equality in numbers of left and right hops, rather than more left or right hops in 10 hops.

**(6) Number of Ways that an Outcome Can Occur:** The notions of permutations and combinations (of Tails and Heads in five coin-tosses) entailed an important conception when reasoning about the likelihoods of outcomes in the sample space in the *Hopping Rabbits* task as just mentioned in the previous subsection. Except Josh, no one seemed to relate the discussion of the most likely and the least likely outcomes to the number of possible ways to get an outcome. By making a connection between the *Bears* task and the five coin-tosses, Josh was, rather unexpectedly, able to make that move earlier than the other students. It was during the ninth teaching episode where I asked students to make predictions for a particular rabbit and then watch the rabbit's five hops ten times using the NetLogo feature of "watching a rabbit." So far, students did many experiments with the computer simulation for various numbers of rabbits, from 10 to 1000, and noted the most and least likely outcomes. However, they did not reason about the likelihood of outcomes in terms of the number of ways to get them, perhaps because the software did not support the distinction between the paths and the outcome, since all the rabbits hopped at the same time, as a block, until the five hops were completed. At the beginning of the task, "watching a rabbit," Josh made a prediction, saying, "one of 1s." When I asked him to explain why he thought that, he reasoned that "it's most likely to land on 1s like in the previous experiments." Then, he further explained that getting HHHHT was not as likely as getting HTHTH since he believed there were "only two ways" to get to "3s" and there were "plenty of ways" to land on 1s. Later, Josh estimated that there might be five different ways for "3s" as a way to convince Maya, who expected the rabbit to land on -3 more than on -1. When in 10 repeated trials the rabbit landed on "3, -3, -1, -1, -3, -3, 1, -3, -1, 3," given that the number of trials was small, Maya's prediction came true. And yet, Josh went on to explain why he still thought there were more ways to land on "1s" than on "3s." Using his strategy from the *Bears* task, he and Caleb listed all the possible ways for each outcome (see Figure 32) while Maya seemed to work on her own by trying different paths, with no systematic way other than the "opposites."

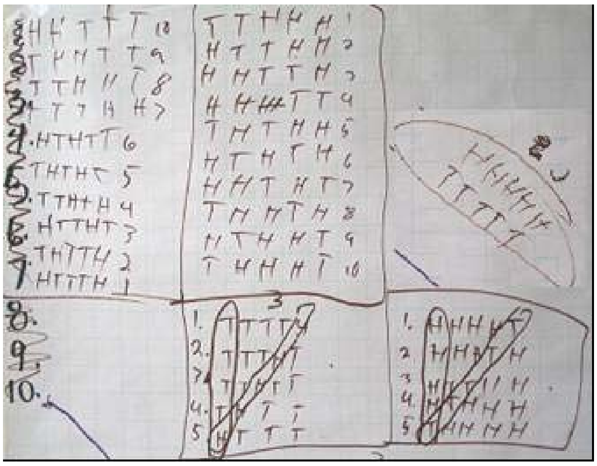


Figure 32. List of all possible ways to get to each final location after five hops (Group 2).

The students in Group 1 also did the same task. Since they still did not have a systematic way to generate all possible ways, except "opposite paths" for negative numbers, like in the *Bears* task, they could only find 10 of the possible ways (out of 20) for 1s (i.e., five in each, -1 and 1), all ten possible ways for "3s," and one possible way for each negative and positive 5. I even gave them some guidance, such as "Can you think of different ways to arrange 3 Rights and 2 Lefts?," but they could not pursue that idea perhaps because they did not develop the notion of combinations yet, based on their responses in the *Bears* task. At the end, students were encouraged to compare the chances, such as P(1s) > P(3s) > P(5s), based on the number of ways in the list of all possible outcomes. For example, Caleb thought that 1s were more likely because they had "more options which means more available." Also, Emily listed the landing on 1s, 3s, and 5s from "the biggest chance" to "the smallest chance." Alicia seemed to continue to reason with the "easier to get" results. For instance, she said that "it was a better chance to get 1 because it is easier to get 1. Because you don't usually get like HHHHH, you usually get like LLRRR." Then I asked her to explain what she meant by "easier." She responded that "if you are flipping a coin, and say you landed on. You don't usually land on HHHHH or TTTTT. You usually land on like TTTHH." With no notion of combinations, it might have been difficult for her to explain what she meant by "easier" (it could be said that it was more likely to get 3 Tails and 2 Heads than to get all Tails or Heads because there were a greater number of ways to get that combination).

Furthermore, Josh and Caleb reasoned with the number of ways to get to an outcome when they had a dispute about whether the rabbits might be more likely to land on 4s than on 2s after 10 hops. Caleb thought that there were more ways to land on 2s than 4s [showing some possible paths on the number line, like HHTTTTHHHH], and less ways to land on 0 [thinking that TTTTTHHHHH and HHHHHTTTTT were the only ways for 0]. Following Caleb's argument, Josh realized that there might be more ways to get to 0 than to 2s and 4s because "you need 5 Tails and 5 Heads." Their problematic then led them to figure out the how many number of ways to get to those outcomes. Once Josh figured out 14 different ways for 0 and 12 different ways for 2. As there were actually 252 different ways for 0 and 210 for 2, I did not expect them to complete their list in this example.

**(7) Quantification of the Probabilities Associated with the Sample Space:** At the beginning of the task, most students tended to quantify only the number of rabbits expected to be on both sides of the starting point (0) after five hops. In doing so, they were encouraged to write their predictions either on the number line at the board or in the table given on the worksheet. Some examples of those estimated quantifications were as follows: Maya: “56 on the left and 44 on the right”; Emily and Alicia: “5 on the right and 5 on the left or 6 on the right and 4 on the left”; Emily: “Around 250 on each side”; Alicia: “Around 5000.” As they conducted simulations with repeated trials and with more rabbits (100, 500, 1000, and 10000) using the NetLogo simulation tool, some of these students were likely to consider the pattern of different likelihoods for each outcome with a symmetry. For instance, when Alicia and Emily made observations from the previous results, such as “more on 5s this time [the number of rabbits increased from 10 to 100],” “very tall,” and “none of them had the same number of rabbits,” their following predictions were often like this: Alicia: “Around 500… More on around 1s and 3s. Just a little bit on 5s.”

Some students, on the other hand, were able to make more specific estimations by reasoning about the likelihoods of outcomes based on the simulation data. Those estimations usually included absolute numbers, such as Caleb: “70 for 1s, 20 for 3s, and 10 for 5s” and Josh: “Around 60 for 1s, around 30 for 3s, and around 10 for 5s.” Furthermore, Josh often tended to refine his predictions based on the previous computer simulation results with the large number of trials and repeated trials even though his first predictions for 100 and 1000 seemed to be proportional to his initial prediction for 10 rabbits, such as “6 for 1s, 3 for 3s, and 1 for 5s” and “close to 60/600 for 1s, close to 30/300 fro 3s, and close to 10/100 for 5s.”

When Josh made his prediction for 500 rabbits, such as “200 for 1s, 150 for 3s, and 50 for 5s,” looking at the results of the simulation, he usually evaluated how close his prediction was and made changes for the next trial. For instance, before adjusting his prediction based on the previous result, he reasoned that “I was more right with 3s. I was really really off with the 1s. Actually I was not. And I was very incredibly amazingly off with 5s." Then, he estimated "290 for 1s, 184 for 3s, and 26 for 5s" with some increase in 1s and 3s and a considerable decrease in 5s. With these predictions he got closer to the next result because his estimations were closer to the theoretical probabilities, i.e., P(1)=P(-1)=0.31, P(3)=P(-3)=0.16, and P(5)=P(-5)=0.03, if they were calculated based on the "frequentist approach to probability", such as P(1s)=0.58, P(3s)=0.37, and P(5s)=0.05.

In addition to the estimations of likelihoods by absolute numbers, students sometimes used fractions to make their predictions. For instance, in the ninth teaching episode, Caleb thought that the rabbit would mostly land on the right side before watching the individual rabbit ten times. Although he failed to recognize the symmetry in the outcomes based on the previous simulation data, the way he reasoned with the fractions was worth looking at it. As seen in Figure 33, he expected that the rabbit might land on 1 and 3 and "kinda" on 5 and would not land on 2 and 4.

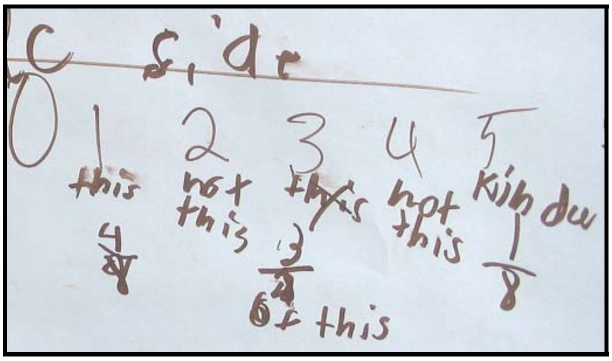



Figure 33. Caleb's estimation of likelihoods of outcomes by fractions.

To explain where he would expect the rabbit mostly on the right side, he began to quantify the likelihoods of possible outcomes as "4/8, 3/8, and 1/8" for landing on 1, 3, and 5 respectively. Those estimations for the likelihoods of landing on 1, 3, and 5 could have approached the theoretical probabilities if they were to apply to the other outcomes symmetrically on the left side of the number line (i.e., 4/8 for both – and + 1, 3/8 for both – and + 3, and 1/8 for both – and + 5).

Finally, Josh quantified the likelihood of outcomes after they came up with the list of all possible ways to get to 1s, 3s, and 5s. By reasoning with the ratios of the number of different ways for 1s, 3s, and 5s to the number of all possible ways in total, he was able to calculate the probabilities as P(1s)=20/32, P(3s)=10/32, and P(5s)=2/32. Note that Josh already had the notion of part-whole relationship for simpler probability situations, as mentioned in the Pre-interview analyses. In order to arrive at the same way of quantifying the probabilities in a somewhat more complex situation, like the binomial rabbit hops, he needed to have a conception of a sample space and development of combinatoric operations, similar to the findings of Piaget and Inhelder (1975). If he separated the number of ways for negative and positive numbers, Josh could have worked out the quantification of probabilities in a binomial distribution using the sample space and the combinatoric operations.

### Task 12: Rolling a Die and Sum of Two Dice

When rolling a single die, some students believed in the equiprobability of outcomes, i.e., "Emily: They are all six sided dice and none of them have 2 fives or 2 sixes" and "Josh: There is 1/6 chance that any of them get picked, 1, 2, 3, 4, 5, 6, because there is only 6 possible outcomes that can happen. It can be any of them [1 through 6]." However, others provided non-probabilistic explanations for the likelihood of each outcome in rolling a die, such as Alicia: "Sometimes it does not land on low numbers" and Caleb: “3 because it always seems to be happening.” Moreover, Caleb and Maya did not seem to believe the pure chance in rolling a die because they thought that “the more you don’t want it, the more you are likely to get it.” Students’ personal beliefs seemed to be persistent also when they were asked to compare the chances of getting different sums, such as 8 vs. 6 and 11 vs. 12. For example, Alicia believed that there was a less chance of getting low numbers and more chance of getting high numbers and thus she picked the larger sum as more likely outcome in both cases. Moreover, Emily thought that the chance of getting a sum of 8 was bigger than the chance of getting a sum of 6 “because people don’t seem to roll a 3 a double number as much as they do non-double number.” Emily then started to reason with the number of ways, but she could not figure out correctly which one would have more ways to get, e.g., “I think you might be able to get 8 in *more ways* than getting 6. But I am not sure.” It seemed that “getting doubles less likely to happen than the other combinations” was a common conception because Josh also thought that all of the outcomes of a sum of two dice had an equal chance, except 2 “because you have to get doubles.” The initial student responses suggested that students tended to consider the probability of each outcome in rolling two dice based on either their personal beliefs about rolling a die or their conceptions about some combinations, i.e., doubles, being hard to get.

Next, to follow up both Emily’s informal conception about the number of ways to get 8 and 6 as a sum of two dice and Josh’ s attempt to figure out the chances for each sum, I asked the students to work together to generate all possible ways to get each sum when rolling two dice. This task was relatively easier than generating all possible ways in the *Hopping Rabbits* task. While listing the opposite paths in the *Hopping Rabbits* task was a common strategy, Emily and Josh did not agree with others in their groups about including both (2,1) and (1, 2) or (3, 2) and (2, 3), in their lists. Emily suggested that the order would not matter when rolling two dice. Similarly, Josh thought that they were not different "unless you are playing a game where it depends on whether 3 is which one is on the left which one is on the right." He also added that it never mattered in board games, based on his personal experiences. This kind of discussion over the order was also reported by Horvath and Lehrer (1998), in which the participants in a 2nd grade classroom study argued using the commutative property of addition (see Chapter 3) not to include both ways.

After listing all possible ways to get each sum in rolling two dice, students tended to expect to get mostly between 6 and 8 if they were to roll two dice 100 times. Even though most of them reasoned with the number of ways to get each outcome, (such as Alex and Emily: "7 because 7 had the most number of ways to be rolled"), Alicia, Caleb, and Maya seemed to fail to understand the purpose of figuring out the number of different ways to get an outcome. While Alicia continued to reason with the earlier belief about getting high numbers vs. low numbers, Caleb believed that there was still equal chance for each because "you don't really know what you'll get." When Maya predicted "3" because she liked it, Josh tried to prove that she was wrong and he quantified the chances of sums. As seen in Figure 34, to show Maya that the chance of getting 3 was quite low (i.e., 1 out of 21), he again calculated the probabilities as the ratio of "the number of ways to get an outcome" to "the number of all possible ways in the sample space." However, note that these probabilities were calculated by ignoring the order in rolling two dice, i.e., no distinction between 1,4 and 4,1 was made by the students.

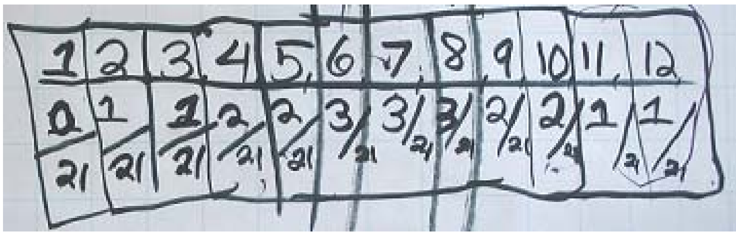


Figure 34. Josh's quantification of outcomes when rolling two dice.

### Task 13: Galton Box

Prior to the last teaching episode in the study, the students completed various tasks, such as the *Split-box* task, the *Multi-level Split-box game*, and the *Hopping Rabbits* task, in which they had both hands-on and computer-based experiences that could be followed up by the idea of the Galton box, in terms of generating data and discussing the likelihood of outcomes in a particular form of distribution. Hence, I asked students to make predictions and run simulations using the NetLogo Galton box model (Wilensky, 2002) which could generate a uniform distribution of balls with one row of pegs and a binomial distribution of balls with five and ten rows of pegs. During the investigations, students usually noticed the resemblance to the particular tasks for the previous teaching episodes, and either they referred to the results in those tasks, or I noted similar patterns in their reasoning as in the previous tasks. In this section, I present the students' reasoning about distributions as they conducted various simulations by changing the number of rows of pegs and the number of balls in the NetLogo Galton box model.

**"Number of Rows = 1" = "Even /Close-to-even":** In this subtask, students' initial predictions highly resembled those made during the *Split-box* task and the *Coin Flipping* task. For instance, Alicia and Emily expected to get equal balls (50-50) in each column because the chance of bouncing right was 50% in the computer simulation.

When they did the simulation repeatedly several times, they noticed the variation in the deviations from 50-50 in the results. Alicia and Emily were quite surprised especially for the result of 36-64 in each column. Similar to their previous reasoning, they expected to get "even" in each column or if not, "around 10 difference" at most. Moreover, Josh made his predictions "close-to-even" by saying that "close to 50-50 with more on right" throughout the task. For others, like Caleb, Maya, and Alex, it seemed that one side was more privileged than the other. Caleb, for instance, predicted "more on left" and Maya expected "more on right" all the time. Also, looking at the results of their simulations, which I recorded on the board, Josh noted that the outcome was "around 50-50 most of the time." Then, I asked them whether they would expect more like 60-40 or more around 50-50? Caleb explained why it would be more around 50-50 by reasoning with the computer feature that controlled the chance of bouncing right in the Galton box, i.e., "Since this [chance of bouncing right in the computer simulation] is 50-50. That means it is going to stay around 50 until we change this like 60 or 70 or something [like they did in the *Hopping Rabbits* simulations]." It seemed that Caleb changed his idea about "everything is possible" as he had claimed in the *Coin Flipping* activity.

**"Increase the Number of Rows" = "Middle most likely / Sides less likely":** When the number of rows of pegs was increased from 1 to 5 with 10 balls, students' initial predictions were again similar to the earlier ones in the *Hopping Rabbits* task. While some students expected more balls in the middle (2$^{nd}$ and 3$^{rd}$ columns in this case) and less on the sides for 5 rows of pegs, others tended to make their predictions in terms of which side (the right or left) would have more balls. Those who predicted 2$^{nd}$ and 3$^{rd}$ columns as most likely places for the balls to fall in usually drew upon their previous experiences. For example, Alicia said that “when we had the experiment, more is going to 1s. When we did the nickels with bunnies and stuff, usually they sort of landed in the middle.” Josh also used the idea of flipping the coin for rabbit hops, but with more specific reasoning about the paths: “It’s like with the coin flipping. When we were flipping the coin, it ended up that most of them landed on 1s, which means that most of them when were doing five hops, which means most of them were heading towards that, we were doing back, forth, back, forth, back, forth, HTHTHT. So this is going to be something like that.” Similar to Josh’s “back-and-forth” idea, Emily predicted that because the pegs would bounce them out everywhere, the balls would usually go like “zigzags” in the middle rather than straight into the sides.

Furthermore, when the number of rows of pegs was increased to 10, the students still expected that the balls would be mostly in the middle columns and less on the sides. However, there was a tendency to predict a larger range for the middle clump in the distribution of balls. In particular, the predictions, like “2 through 7” and “4 though 7,” predicted by Alicia and Emily, respectively, might suggest that they expected the shape to become more “spread-out” and less “tall.” This possible claim is very relevant to probability distributions, such as a normal distribution, in which the area under the density curve is always equal to 1. Hence, when we look at a family of normal distributions, their shape will be “tall and skinny” or “short and wide” depending on the variation of data across the possible outcomes (Kazak & Confrey, 2004). To illustrate the conservation of area in a probability distribution, the binomial distributions of five and ten rabbit-hops were simulated and displayed as histograms in Figure 35. As suggested by Emily and Alicia’s predictions, the shape of the distribution for five hops was tall and skinny whereas the shape of the distribution for ten hops turned out to be more spread-out and shorter in the middle because of the tendency of preservation of the area under the bars.

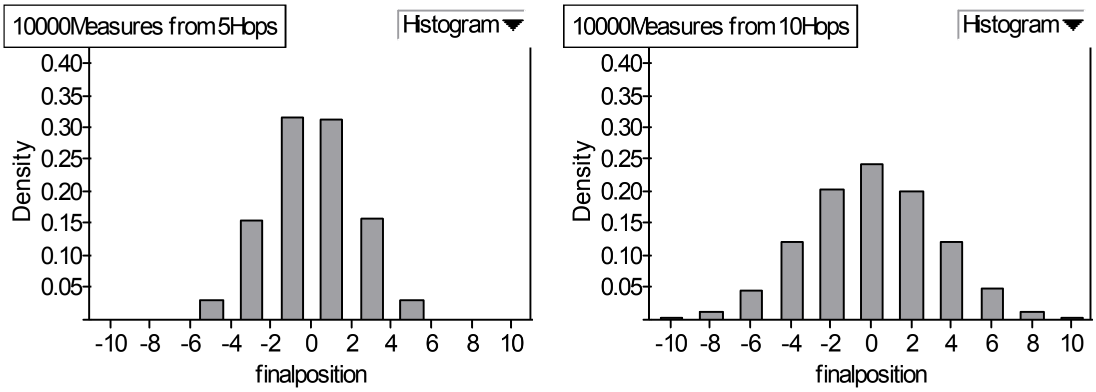



Figure 35. A simulation of five hops and ten hops for 10,000 rabbits.

**"Increase the Number of Balls" = "More scattered / More in each column":** As the number of balls increased from 10 to 100 in the Galton box with five rows of pegs, students still expected "more in the middle and less on the sides" and "more balls in each column" than there used to be. For example, Emily said that "More on 1, 2, 3 still, maybe some in the 4, but there will be more than one on 0 and 5 probably...because there is a bigger number of balls together. Maybe around 10." Then, Alicia added that "I agree with her. There are more balls and they'll *scatter around more*." I also noted that Emily used a version of the term "distribution" in her explanation, i.e., "More on the sides and more in the middle because there is more of them to *distribute* to everywhere." Caleb seemed to expect "more on each side" as proportional to the previous simulation with 10 balls when he predicted 60 on the right side and 40 on the left side and said "since we didn't change the number of rows, I am keeping my answer from the previous one [6-4]."

The students, then, conducted a number of simulations with 100 balls repeatedly to make conjectures about the most likely outcomes when the number of rows of pegs was five. After their investigations, I asked students, “Why do we have more on 2 and 3? And few on the sides (0 and 5)?” For the most likely outcomes, such as columns 2 and 3, they usually drew upon their observations about how the balls moved down in “zigzags” and “back-and-forth” in the middle part. One way to justify the most and least likely places was to turn on the “shade-path” feature in the NetLogo simulation and watch the changing shades of color based on the frequency of balls passing through that spot (see Figure 36). For instance, in Group 1, students began to watch as the balls moved down and the paths were shaded:

*Emily*: How come it’s turning blue?
*Sibel*: What do you notice?
*Emily*: It’s turning blue.
*Sibel*: Why?
*Emily*: Because lots of them going that way.
*Alicia*: And I was right. They usually go *zigzag* around here.
*Sibel*: So, where do you expect more blue?
*Alicia*: Around here [showing the middle part of the rows of pegs].
[They are watching the balls, the numbers of balls in the columns, and the shading]
*Emily*: See, it’s turning really dark blue at the top because most of them coming right here [the first row of pegs at the top] and they go either to that side [left] or to that side [right]. And they go down here [to the middle] [showing “zigzags”] and rest of there [to the sides].

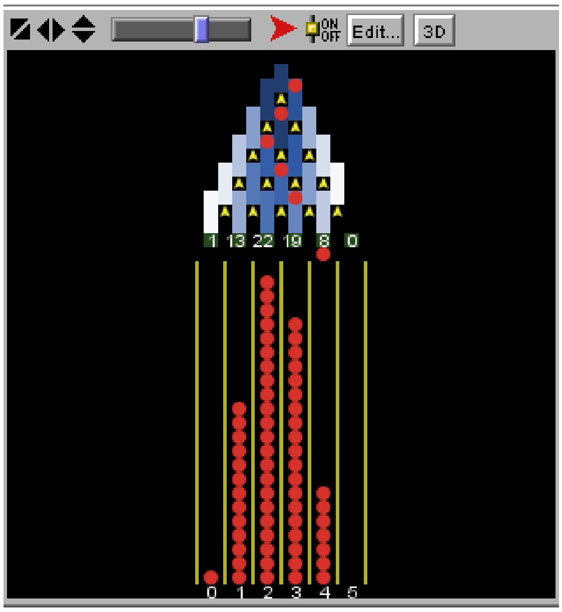



Figure 36. The NetLogo Galton box model interface showing the “shade-path” feature.

As seen in this excerpt, students were able to watch the paths of the balls as they accumulated in the columns at the bottom. It was relatively easy for all students to conjecture that the shade of the color got darker on particular areas as the balls passed through those areas more frequently. They could also justify their previous predictions about the paths of the balls, such as Alicia’s “zigzag” idea. Students tended to use this feature of the computer simulation when they ran simulations by changing the chance of bouncing right to 75% and to 25%. They initially made predictions about where the shade might get “darker and darker” on the triangular (rows of pegs) area. For example, for the right skewed distribution of balls, they expected the region towards the right side of the triangular area to get darker. They seemed to make use of the shading feature to reason about the most likely and the least likely outcomes [the numbered columns at the bottom] for the skewed distributions also.

Furthermore, in Group 2, there was an interesting discussion about the less likely outcomes, such as columns 0 and 5:

*Sibel*: Why do we have few on the sides?
*Maya*: It's not possible that they'd go [showing a straight path to the sides on the computer screen].
*Sibel*: Why is that not possible?
*Maya*: It's not possible for it to go that down there like 100 times.
*Josh*: It actually is. It's just not very likely.
*Sibel*: What do you mean by "not very likely"?
*Josh*: It's not likely at all. In other words, it's imp...not impossible. It's pretty close to impossible.
*Maya*: That's what I was trying to say.
*Josh*: It had to be TTTTT…

In this discussion, it seemed that both Maya and Josh did not expect the balls to go down straight to the sides so often, like "100 times." In order to make the same argument, students were trying to negotiate a language to justify that some outcomes might be less likely to occur. Note that Josh began to explain the less likely outcomes by switching to the five coin-flips used for five rabbit-hops. Caleb also suggested that HHHHH was "nearly impossible."

Building upon the close link between the *Galton box* model and the *Hopping Rabbits* task, Josh anticipated that if there were five rows in the Galton box, then there would be 32 different ways that the ball could fall down. Then, he showed some possible paths through the pegs on the Galton box. After Josh's conjecture about the different ways to get to each outcome, which I did not anticipate earlier as something that would occur in this task, I asked Maya and Caleb about what they thought about his idea. Caleb seemed to see different paths to get to the same column when he said "it won't always be HHTTH. It could be HHHTT. I can't really agree with him, but I can't tell he's wrong," and yet he could not justify Josh's conjecture further. Moreover, Josh believed that there were more ways to get to 2 and 3 by reasoning that a path like HTHTH was more likely to happen than a path like HHHHT. When I asked him to explain his reasoning, he said, "see because Heads and Tails are both 50%. There is a 50% of a chance that either of them are going to get flipped or whatever. So, it's not very likely that you are going to get something like HHHHT because there is so many Heads in it, you know compared to the Tails. Now, HHHTT is pretty close to even." His last explanation suggested that he actually was looking at the number of Tails or Heads in a sequence to make a judgment about the more likely paths. Similar to the earlier expectations about "even" and "close-to-even" numbers of Tails and Heads with many coin-flips, he reasoned that if there were about equal number of Heads and Tails in a path, then it would be more likely to occur. It could be also possible that the idea of "it's going back and forth, back and forth" might apply to having an almost equal number of right and left, or Tails and Heads, in the likely paths. His reasoning about the number of Heads and Tails in a path could also be related to the notion of different combinations, such as 4 Heads and 1 Tails, 3 Heads and 2 Tails, and so on. If so, it was true that getting 3H2T would be more likely than getting 4H1T because there were more ways to get the former combination.

## Summary

My initial conjecture in designing this teaching experiment study was that the notion of distribution was a conceptual link between chance and data. In particular, the design and sequence of the thirteen tasks were driven by that conjecture. Therefore, the initial tasks focused on the development of students' informal reasoning about distributions by means of qualitative descriptions of such features as the middle, spread, and shape. For example, students paid attention to the groups and the different group patterns in the distributions (i.e., the natural distributions in Task 1 and the distributions of chips in Task 2) and in turn developed ways to describe them with their informal language (i.e., "piled up," "scatter out," "pushed off that way," and "packed together"). Then, they began to reason about distributions quantitatively by using the notion of density (i.e., "Up here there is not that many. They are more *spread out*. But in here they all are *packed up tightly*…"). Moreover, their investigations of the certain effects, such as the height and the number of chips, on the distribution of chips led them to generate their own games in which different scores were assigned to the places for chips to land closer or further from the middle point.

In addition, students tended to conduct their own investigations in order to make precise predictions for the distribution of marbles in the split-box, such as how many more marbles might be in the right slot. Students also began to develop conjectures about the results of equiprobable events and to generalize them into similar situations. For example, by dropping all marbles in the middle or evenly from each slant at the same time), the "middle" conjecture was developed to predict "even" or "close-to-even" results in the split-box. Then, the "middle" conjecture was generalized to predict the numbers of Heads and Tails as "even" or "close-to-even" with the idea of "split in the middle" in the coin-flips.

In the tasks devoted to the discussion of binomial distributions, such as the *Hopping Rabbits* task and the *Galton Box* task, students first developed the idea of sample space (all possible outcomes in a chance event). Then, through the simulations, they discussed the most likely and least likely outcomes based on the qualitative aspects of distribution, such as the spread, symmetry, skewness, and middle. Furthermore, the student-generated inscriptions supported a richer understanding of these binomial distributions as students often used the paths to describe the easy and hard to get outcomes (i.e., "It's harder to get there (-/+ 5) because they usually go like [RRLLR] and less like [RRRRR]"). As conjectured initially, students' construction of ideas of permutations and combinations supported their reasoning about the number of ways when discussing the most likely and least likely outcomes, such as in rolling two dice, binomial rabbit-hops, and the Galton box model. By conducting a number of computer simulations in the *Hopping Rabbits* task and the *Galton Box* task, students began to estimate empirical probabilities when they revised their predictions for the outcomes, such as the number of rabbits or balls on a particular location, on the basis of previous simulation results. Eventually, one student (Josh) developed an understanding of theoretical probabilities (probability as a ratio of the number of all possible ways to get a particular outcome to the number of all possible outcomes) by constructing on his conceptions of sample space and combinatoric operations.

The emerging conceptual trajectory for developing an understanding of probability concepts through reasoning about distributions both qualitatively and quantitatively is further discussed in the final chapter with respect to the research questions. The next chapter examines the results from the post-interviews. Also, Chapter 8 discusses the major findings of the study from data collected during the pre- and post-interviews and the teaching experiment study.

# CHAPTER 7

# ANALYSIS OF POST-INTERVIEWS

In this section, I discuss the participants' responses to each interview tasks (the same as the ones used in the pre-interviews) and their reasoning processes during the post-interviews. The readers should note that it was not the intention to quantify the post-interview results by means of a comparative analysis of "amount of ideas learned" by the students over the course of seven-week teaching experiment. Rather, the purpose of conducting post-interviews was to document the kinds of changes in their understanding of probabilistic concepts and reasoning after engaging in reasoning about distributions in chance situations through a variety of tasks which involved the notion of equiprobability, comparing the likelihoods of outcomes, listing the sample space, and generating all of the different possible ways to get each outcome in the sample space.

## Post-Interview Task 1: Channels

In the post-interviews, most students, such as Alicia, Emily, Josh, and Maya, identified the channel systems A and D (see Figure 37), in which the marble would equally be likely to come out of exit 1 or exit 2, by using probabilistic reasoning, rather than causal, deterministic, or mechanical explanations. The channel system B remained difficult for many students, i.e., Alicia, Alex, Caleb, and Maya, to understand and to justify the equiprobability in it. Moreover, all students, except Alex, provided probabilistic reasoning to explain that exit 1 was more likely than exit 2 for the marble to come out in the channel system C. However, only Emily and Josh could reason probabilistically about the channel system E whereas the others, except Maya, thought that middle exit was more likely, but their explanations included either causal reasoning or no justification.  Next, I will discuss each student's responses for identifying the figures with equiprobability and with non-equiprobability.

1) This is type of a guessing game. There are five figures showing the different channels (see A, B, C, D, and E). Suppose we place a marble at the top and let it drop many times one after the other, and then it will come out at the bottom in one of these numbered exits.

Circle the figures where you think that the marble is as likely to come out of exit 2 as exit 1. Explain how you got that answer.

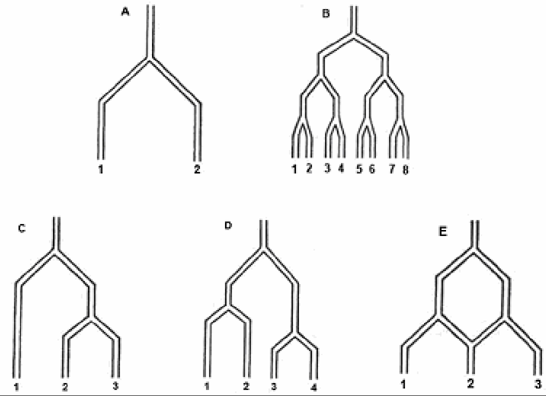



Figure 37. Post-Interview Task 1: Channels.

As in the pre-interview, Alicia provided probabilistic reasoning to explain the equiprobability in figures A and D.  She tended to use "50-50" notion to justify her response.  For instance, she believed that the chances were 50-50 in figure A because "all you have is two" and said "if you have 50-50 chance, it's half and half. If there is 10 marbles, since there is 50-50 chance there will be 5 marbles in each."  However, she failed to recognize equal probabilities for each exit in figure B.  Since there were many exits in this one, she thought that if there were 100 marbles, they would "usually spread out" randomly, such as "20 in 1, there could be 30 in 2, then there could be 10 in 3, 10 in 4, then 5 in 5, another 5 in 6, you might get 10 in 7 and you might get another 10 in 8." Alicia was able to provide probabilistic reasoning when she said that exit 1 was more likely in figure C by noticing the two options for a marble to go in on the right side: “if it goes to the left, then it will go straight down. If it goes the other way [right], it may go to [exit] 3 randomly.” In figure E, when Alicia recognized that exit 2 had more chances because “balls go to the middle mostly,” she seemed to draw upon the outcomes in the Galton box model they studied during the teaching experiment. However, she could not justify her response with the number of possible ways that might lead to exit 2.

Alex also seemed to be using the previous results, such as the simulations in the Galton box model, because he always expected to have the marble mostly in the middle channel without any justification other than saying, “because that’s what happened when we did the experiments.” However, he could not explain why he thought that these channels looked like the previous experiments and why the marbles mostly went to the middle in those experiments. Except that in figure A in which there were only two channels, he thought that exit 2 had more chance, but with no explanation.

Similar to her responses in the pre-interview, Emily thought that there was an equal chance of getting the marble out of exit 1 and exit 2 in figures A, B, and D on the basis of the idea that the marble could go either way when it arrived at the fork before splitting to 1 or 2. For figure B, due to the many channels, equiprobability was difficult to be noticed by some students. However, without any prompt, Emily tended to simplify the situation based on the questions asked when she said that “although there are many compartment things, we are only talking about 1 and 2 and those are both equal.” Unlike her deterministic explanation for figure C in the pre-interview, Emily used a probabilistic reasoning to justify that exit 1 had more chance than exit 2: “it’s just because in this one it can either go to 2 or 3, so it’s going to be more likely to go down at 1.” Again, in figure E, she used the number of ways to get to each exit to explain why exit 1 was not as likely as exit 2 and said that "there are two ways to get to 2 because say the little marble went down here [left] and then it went down here, then it could go in to 1 or 2. But say the little marble went down here, it's going to go in to 2 or 3 which means there is two ways to go on to the 2."

Caleb's reasoning in this task usually remained causal and deterministic as in the pre-interviews. When he argued that figure A had equal chances for exit 1 and exit 2, his reasoning, i.e. "because you can't really tell. Because if you drop it, it could go this way or if you drop it, it could go that way," seemed to be more based on the outcome approach (Konold et al., 1993) in a sense that "anything could happen." Even though he initially thought that the exits were all equally likely in figure B, between exit 1 and exit 2, he chose 2 as more likely outcome based on causal and mechanistic explanations including where the ball might hit and then which direction it might go. Again based on deterministic and causal reasoning, in figure D, he believed that exit 1 was more likely based on the assumption that "once the ball goes to the left, it still wants to go to the left." However, Caleb noticed the case of unequal probabilities in figure C by reasoning that the chances split when the marble went to the right side: "this is easier [to go to exit 1] because if it goes down here [right], it could go to either one of these [exits 2 and 3]." Although Caleb identified that exit 2 was more likely than the others in figure E, he again drew upon some causal explanations, such as "because it is going down here [left] and here [right] and this one [right channel] is more of a hill than that one [left channel]. So it goes to more down to 2," rather than the number of possible ways to get to exit 2.

Josh seemed to be the one whose responses in the task shifted the most towards probabilistic reasoning from purely deterministic and mechanical reasoning in the pre-interview. For the figures with equiprobability, he always used the same reasoning, "it could go either way." For instance, in figure A, he explained his response by making a connection to the *Split-box* task in the teaching experiment: "This one obviously because it's you know that's basically the same thing as the box you brought in because there is one separator and then you know it can *go either way*." For figure C, he could even calculate the chances for each exit using the multiplication of probabilities: "You know there is a chance of going down to 1 that's like 50% chance. And then, the other 50, but then right here, it is more like of a 25% chance that it's going to be in 2 and 25% of chance that it's going to be in 3. [*Why?*] Because it's 50-25-25." Moreover, Josh seemed to understand the addition of probabilities when he quantified the chances for each exit in figure E. His explanation was as follows:

> *Josh*: If the one right here, you know the ball could go to 2 or 1. But let's just say one over here, then it could go to 2 or 3. So, this [exit 1] is more like of a 25% chance, this [exit 2] is 50, and this one [exit 3] is 25.
> *Sibel*: Why this has a 50% chance?
> *Josh*: Because no matter which way it goes off of the first one [the fork], it has still chance of going to 2. But if it goes this way [left], it doesn't have a chance of going to 3. If it goes this way [right], it doesn't have a chance of going to 1.

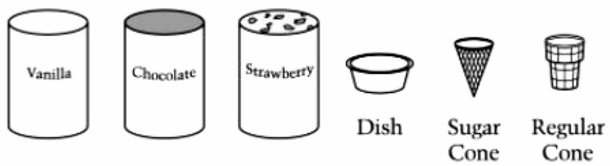
Maya's responses in the post-interview were also more consistent through the task and more likely to provide probabilistic reasoning in her explanations. For example, in figures A and D with equiprobability, she explained her response by saying "because if it drops right here, it could go either way." However, like some other students, Maya had difficulty in generalizing the equiprobability notion in figures A and D to figure B, which was more complex. Hence, she thought that the chances were not equal for exits 1 and 2 “because it could go into these [the other exits].” On the other hand, Maya overgeneralized the equiprobability when she focused on the left side of the channels in figure E and ignored the connection to exit 2 from the left side. Thus, she thought that both 1 and 2 were equally likely. However, when she thought about different paths, one to the left and one to the right, in figure C, she figured that “1 would have a better chance because it can go just go down here [to exit 1] or here [on the right] it might go to 3 or it might go to 2.”

Table 12. Students’ responses to the “Channels” task in pre- and post-interviews (I = Incorrect, C = Correct)

| | **Channel Systems** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **A** | | **B** | | **C** | | **D** | | **E** | |
| | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post |
| **Alex** | I | I | I | I | I | I | I | I | I | I |
| **Alicia** | C | C | I | I | C | C | C | C | I | C |
| **Caleb** | C | C | C | I | I | C | I | I | I | I |
| **Emily** | C | C | C | C | I | C | C | C | C | C |
| **Maya** | C | C | I | I | C | C | I | C | I | I |
| **Josh** | C | C | I | C | I | C | I | C | C | C |

In this task, the students’ responses during the post-interviews could be summarized in terms of correct (C) and incorrect (I) compared to those in the pre-interviews as seen in Table 12. More specifically, students’ correct responses in the post-interviews suggested that more students, such as Emily, Josh, Alicia, and Maya, tended to understand the ideas of equiprobability, multiplication of and addition of probabilities (both visual and quantitative) and to reason probabilistically. However, some of them still used some deterministic and causal reasoning occasionally to explain their answers. Also, it seemed that students sometimes drew upon the previous tasks in the teaching experiment. If they had a good understanding of the experiments to which they referred, they were more likely to use the idea in a supportive way to justify their responses. For example, the reason that the balls went to the middle mostly in the Galton box was due to the fact that there were more numbers of possible ways to go to the middle columns at the bottom. Similarly, in figure E, there were two possible ways for a marble to come out of exit 2 whereas there was only one way to go to the exits 1 and 3.

**Post-Interview Task 2: Ice-Cream**




2) Jan's Snack Shop has 3 flavors of ice cream: vanilla, chocolate, and strawberry. The ice cream can be served in a dish, a sugar cone, or a regular cone.

- There are 9 people who choose 1 dip of ice cream in a dish, or in a sugar cone, or in a regular cone, and all of their choices are different. List or show the 9 different choices.
- Could another person have a choice that is different from one of these 9 choices? Why or why not?

Figure 38. Post-Interview Task 2: Ice-Cream.

In the post-interviews, every student was able to find all nine different choices by figuring out all possible combinations of having one dip of ice cream in a container. They all seemed to come up with a systematic way of showing these nine choices, such as list, paths, and drawings. Even though Alex and Maya failed to generate all of the possibilities in the pre-interviews, they now seemed to understand the sample space in the problem to be able to show the combinations of one dip of ice cream in a container. All of them also were positive that there was no other possibility different than the nine choices they generated with three flavors of ice cream in three different containers. Some examples of students' responses were as follows:

> *Alicia*: There are 3 things and you can mix them 3 times because there are 3 of each.
> *Alex*: No. If there is more cones that were different.
> *Maya*: No, I don't think so. They don't have chocolate syrup, and you can't get two types of ice cream. So, I guess there is no other way.
> *Josh*: No, because there are only 3 different choices for each one. So, 3 different choices can go for each flavor. But that's it. There aren't. If there are 4 different choices, then there'd be like 12, but there aren't 4 choices. So.

Note that Maya's reasoning during the post-interview was based on the sample space in the task, rather than irrelevant possibilities based on her personal experiences that she expressed in the pre-interview. Moreover, Josh's way to explain the number of all possible choices you could make suggested that he could use the basic counting principle, that is, the multiplication rule (i.e. If there are m ways to do one selection, and n ways to do another, then there are m*n ways of doing both).

**Post-Interview Task 3: Swim Team**

3) There are 3 fifth graders and 2 sixth graders on the swim team. Everyone's name is put in a hat and the captain is chosen by picking one name. What are the chances that the captain will be a fifth grader?

A) 1 out of 5
B) 1 out of 3
C) 3 out of 5
D) 2 out of 3

Explain how you got that answer.

Figure 39. Post-Interview Task 3: Swim Team.

Similar to the responses in the pre-interviews, Emily and Josh both were able to quantify the probability of a single event in the problem ("3 out of 5") as the ratio of the number of favorable cases to the number of all possible outcomes (Classical definition of probability). In the post-interviews, Alicia and Caleb also calculated the probability as "3 out of 5" with the explanations below:

> *Alicia*: Because there are 5 kids and 3 fifth graders and 5th graders has more chance.
> *Caleb*: 5th graders have 3 out of 5 chance of getting picked since they are more than the 6th graders.

In Caleb's response, it was not clear how he decided to have 3 out of 5. Then, I asked him what if there were one 5th grader and two 6th graders. His response was "1 out of 3. He doesn't have as much chance as 6th graders." It seemed that he could calculate the probability, like Emily and Josh, but his explanation was vague and qualitative rather than "Emily: Because 3 5th graders plus 2 6th graders equals 5 students except there is 3 5th graders. So, 3 out of 5." Alex's explanation for his response "3 out of 5" was the same as Caleb's, such as "because there is more 5th graders than 6th graders," but he could not justify the calculation. Only Maya seemed to fail to calculate the probability correctly in this problem because she thought that it was "1 out of 3, because there are 3 5th graders and it could be 1 out of 3…there is 3 of them, you get to choose only one." Similar to her reasoning in the pre-interview, Maya failed to understand the whole when considering the favorable cases.

## Post-Interview Task 4: Stickers

<table>
<tr><th>Stickers</th><th>Number</th></tr>
<tr><td>Red</td><td>|||</td></tr>
<tr><td>Blue</td><td>||||</td></tr>
<tr><td>Yellow</td><td>||</td></tr>
<tr><td>Green</td><td>𝍸 ||</td></tr>
</table>

4) The 16 stickers listed above are placed in a box. If one sticker is drawn from the box, which color is it most likely to be?

A) Red
B) Blue
C) Yellow
D) Green

Figure 40. Post-Interview Task 4: Stickers.

Like in the pre-interviews, all of the students said that the green sticker was the most likely one to be picked because there were more of them in the box than all of the other colors. Furthermore, Josh justified his response by calculating the probability of picking a green sticker, i.e., "Because there are more greens in there. So the chances are higher that the green going to be drawn. There is 7 out of 16 chances that it is going to be green." Even though some of these students could not reason about the probability in the previous task (Task 3: Swim Team) with no understanding of the part-whole relationship, they seemed to have a conception of probability as a function of the given frequencies which could develop earlier than the former understanding (Piaget & Inhelder, 1975).

## Post-Interview Task 5: Marbles

5) Steve was asked to pick two marbles from a bag of yellow marbles and blue marbles. One possible result was one yellow marble first and one blue marble second. He wrote this result in the table below. List all of the other possible results that Steve could get.

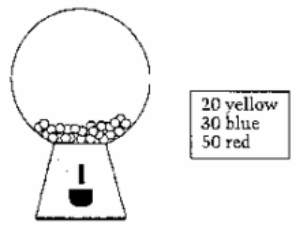
| First Marble | Second Marble |
|---|---|
| y | b |
| | |
| | |
| | |

y stands for one yellow marble.
b stands for one blue marble.

Figure 41. Post-Interview Task 5: Marbles.

More students, such as Emily, Alicia, Caleb, and Josh, could complete the list of sample space (b and y, b and b, y and y) by considering all possible outcomes of drawing two marbles from the bag of yellow and blue marbles. Emily, Caleb, and Josh, also, were successful in this task in the pre-interviews, except Alicia. In the post-interview, Alicia was positive that she listed all the possible results, saying that "That's all because there is only 1st and 2nd." Note that she was the only one who mentioned the order in the arrangement when she made her explanation about the permutations she listed. Moreover, Josh seemed to make a connection to the tasks they did in the teaching experiment when he noted that "it's kind of like what I was doing with X's and stuff, with the Bears and Rabbits." The other students, Alex and Maya still had trouble in listing the sample space because they could only list one of them, "b and y". They both used the "opposites" strategy that was often used in the *Bears* task and the *Hopping Rabbits* task when generating all the possible outcomes. However, Piaget and Inhelder (1975) found when children were asked which outcome would most likely to happen if they were to draw all the marbles two at a time from a bag in which there was equal number of blue and red marbles, they developmentally could first think of the same-color pairs, such as BB and RR, in the first stage (ages 5-6) and the mixed results, i.e., RB and BR, later in the second stage (ages 6-10). Then, maybe Maya and Alex tended to think of the mixed outcome, rather than the same-color results, because one of the mixed results was already given in the problem.

**Post-Interview Task 6: Gumballs**




6) The gum ball machine has 100 gum balls; 20 are yellow, 30 are blue, and 50 are red. The gum balls are well mixed inside the machine.

Jenny gets 10 gum balls from this machine.

What is your best prediction of the number that will be red?

Answer:____________________ gum balls

Explain why you chose this number.

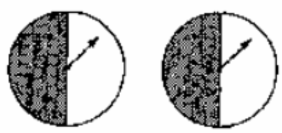
Figure 42. Post-Interview Task 6: Gumballs.

Most students predicted 5 red gumballs in the post-interview. Of those, only one seemed to use proportional reasoning. For instance, Josh explained his response by saying, "It's sort of converting the 100 to 10 which makes the 50 to 5." There were others who used the quantities in the mixture to predict 5, but their explanations were insufficient to claim a proportional reasoning:

> *Emily*: Since 5 and 5 equals 10, then there is 50 red gumballs. I just stick out the 0 and I got 5 and have 2 yellow and 3 blue. There will be 5 reds.
>
> *Caleb*: Because 5 is kind of. 50 resembles 5 and 30 resembles 3 and 20 kind of resembles 2. So, 2 plus 3 is 5. Then, 5 reds, 2 yellows, and 3 blues.

Emily's reasoning was similar to Josh's response in the pre-interview, i.e., "take all the zeros off." In Caleb's explanation, one could argue that "50 resembles 5" might mean 50:100 as the same as 5:10. However, there was not enough evidence to claim that. Similar to their initial responses, Alicia and Alex answered 5 with explanation that there were more red gumballs. On the other hand, Maya's response in the post-interview, "3 red gumballs," was quite different because her prediction was "7 red gumballs" in the pre-interview. She said that she would expect less red gumballs (3R, 4B, 3Y) this time "because that's what happened to me and it can happen to that girl." She explained that there were a lot of pink gumballs in the machine at the mall, but she got 1 red and 1 yellow even though she thought she could get a pink one as there were more pink gumballs. Her reasoning could be explained by the "availability heuristic" (Kahneman & Tversky, 1973), which implies that the availability of her experience at the mall biased her reasoning to make a prediction about the likelihood of getting a red gumball.

**Post-Interview Task 7: Spinners**



7) The two fair spinners shown above are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once. James thinks he has a 50-50 chance of winning. Do you agree? Justify your answer.

Figure 43. Post-Interview Task 7: Spinners.

In the post-interviews, most students still seemed to agree with "50-50" chance of winning in this task. Alicia, Emily, and Caleb reasoned that the spinners were "half and half" (equal areas of white and black on the spinners) whereas Alex, like his earlier reasoning, said that "there are different reasons. Because it depends on how much you spin around. How many times…I mean, like all the way around. Because if it goes all the way, it can end up in the same place." As Josh was able to reason with the sample space including "bb, ww, bw, and wb," he said, "[James] has a 25-75 chance of winning. [*Why?*] Well because there is four different outcomes, there is ww, bb, bw, wb. So it's kind of 1/4=0.25 of chance." Maya also did not think that Steve had a 50-50 chance of winning. She realized that "that's not a fair game" because she thought that it was hard to get both on black since one might land on white and then he would lose the game. Her response sounded reasonable although it was not based on explicit normative probabilistic reasoning, which could involve either the sample space, like Josh did, or the multiplication principle. When I asked her some probe questions, she began to quantify the chance of winning the game:

> *Sibel*: So, you don't think James had 50-50 chance of winning?
> *Maya*: No, not anymore.
> *Sibel*: Do you think he has more chance or less chance of winning?
> *Maya*: His chance of losing is probably 60 and his chance of winning is probably 40.
> *Sibel*: Why?
> *Maya*: Because that arrow could be wrecked, or the wind could blow it or something. I think he has 60% of loosing because some people try to mess up arrows.

Her prediction about the chance of loosing (60%) was quite close to the actual probability of 75%. However, her response at the end was indeed based on causal reasoning rather than a mathematical justification which Josh provided.

Next, the final chapter further discusses the results from the pre- and post-interviews and the teaching experiment with respect to the research questions. The conceptual corridor of possible opportunities, landmark conceptions, and obstacles for reasoning about distributions in probability situations is also delineated. Moreover, the final chapter states the limitations of the study and its implications for research and practice in mathematics education, and future research.

# CHAPTER 8

# DISCUSSION AND CONCLUSIONS

Previous studies separately examined (1) how students arrive at probabilistic judgments and what strategies they employ; (2) how children develop the concept of probability; and (3) the characteristics of students' probabilistic reasoning. However, this research study suggests a holistic approach that brings together the discussions of students' probabilistic reasoning, misconceptions, beliefs, and their development of understandings of probability concepts. To do so, this study documents the landmark conceptions and obstacles students have as well as opportunities to support students' development of probabilistic concepts (e.g., equiprobability, sample space, combinations and permutations, the law of large numbers, empirical probability, and theoretical probability[1]) and probabilistic reasoning. Hence, the goal of this study was to characterize a conceptual corridor involving possible conceptual trajectories taken by students based on their conceptions of probability and reasoning about distributions in chance events by the design of a sequence of tasks.

The data analysis chapters (5, 6, and 7) reported on the students' understanding of the fundamental probability concepts and probabilistic reasoning in the pre- and post-interviews and their development of probability concepts and reasoning about distributions in chance events in the teaching experiment. In this chapter, I first revisit the research questions and discuss the findings of the study to characterize the conceptual corridor for reasoning about distributions in chance situations. In addition, I present limitations of the study, its implications for research and practice in mathematics education, and directions for future research.

[1] See the definitions of these terms in Appendix E.

## Answers to the Research Questions: Conceptual Corridor

The main research question of the study was: How do students develop reasoning about distributions when engaging in explorations of chance situations through a sequence of tasks in which students were asked to provide predictions and explanations during the experiments and simulations with objects, physical apparatus, and computer environment? To unpack this larger question, I considered the following four supporting questions in articulating the conceptual corridor for reasoning about distributions in chance situations:

1) What are the students' prior knowledge about probability concepts and probabilistic reasoning?
2) What kinds of informal knowledge and strategies can serve as starting points?
3) What are the conceptual trajectories that students take during the teaching experiment?
4) What are the resources (learned ideas) students bring into understanding of probabilistic concepts and reasoning?

Next, the answers to these supporting questions are discussed based on the results presented in Chapters 5, 6, and 7. Then, I model the conceptual corridor which was the goal of this design study.

### Answer to the First Supporting Question

The pre-interview findings addressed the first supporting question by examining students' prior understanding of probability concepts (e.g., equiprobability, sample space, multiplication and addition principles of probabilities, combinations, permutations, classical view of probability, probability as a function of given frequencies, probability of a compound event) and the ways students reasoned in chance situations. The results were discussed with details in Chapter 5 for each interview task.

First, in Task 1, "Channels," a majority of the students did not have intuitive ideas about equiprobability and multiplication and addition principles of probability (if they did, it was limited to physical or visual representations) prior to the teaching experiment. Students mostly were able to identify the equiprobability in the channel system with two routes (figure A, see Appendix A), e.g., "it might go this way or that way." As the number of equiprobable exits increased, such as figure D and figure B (see Appendix A) with four and eight exits respectively, most students failed to recognize the equally likely paths to each exit. Justifying the non-equiprobable routes also seemed to be challenging because students needed to understand the multiplication and addition of probabilities either visually (number of paths to exits) or mathematically (calculation of probabilities). The main reason for students' poor responses to justifying both equiprobable and non equiprobable routes in complex figures was that they mainly relied on mechanistic or deterministic reasoning. That is, most of the students focused on deterministic interpretation of the situation, such as "…if it goes there, it will hit this and it keeps going down there [R]" and "…because it is tilting towards [exit] number 2."

Second, most of the students demonstrated a systematic way to show all possible combinations of three flavors of ice cream served in three different kinds of container. These included paths from each flavor of ice cream to each kind of container on the picture given in the task, a pictorial representation of each combination of a dip of ice cream and a container, and an ordered list of choices by keeping fixed either the flavor of ice cream or the container type and changing the other in an orderly way. Also, students' understanding of permutations was assessed in Task 6, "Marbles," in which students needed to consider all possible outcomes of picking two marbles from a bag that consisted of yellow and blue marbles. Only three of these students entered the teaching experiment with an understanding that their approach identified all possible combinations of two elements (the same color marbles and the mixed color marbles). This finding was consistent with children's development of combinatoric operations, implying that the discovery of permutations followed the development of the idea of combinations (Piaget & Inhelder, 1975). In the teaching experiment, these two ideas (combinations and permutations) reemerged when students began to reason with paths in the multi-level split-box game (paths of left and right turns on the counters) and in the *Hopping Rabbits* task (paths of left and right hops). Moreover, the *Bears* task (prior to the *Hopping Rabbits* task) supported students' development of a systematic way to figure out all different possible ways to arrange red and blue bears in a row of five bears.

Third, one third of the students were able to represent the probability of a single event as a ratio of the number of favorable cases to the number of all possible outcomes whereas the rest failed to recognize the relationship between the part and the whole in quantifying the chance. And yet, all students had an intuitive conception of probability as a function of the given frequencies which, according to Piaget and Inhelder (1975), developed earlier than probability as a ratio of the number of favorable cases to the number of all possible outcomes.

Fourth, some students made a prediction about a sample in proportion to the population, but most of them did not explicitly justify their response with the percentage of the red gumballs in the mixture, a precise justification using proportional reasoning. Then, others predicted the values that they thought reflected the distribution in the population. For instance, they predicted slightly more than half for red gumballs since there were "more red gumballs" in the mixture, rather than focusing on the relative ratios of the colors.

Last, none of the students entered the study with an understanding of the probability of a compound event composed of two independent events in an experiment. Hence, their reasoning was mainly based on either causality or the notion of "50-50" in each independent event, rather than derived from the sample space or the multiplication principle of probability. The main reason I think students reasoned with "50-50" chance in each independent event was because they failed to establish whether the compound event was also equiprobable or not. More specifically, students thought that because each independent event (i.e., getting a "black" on a spinner with two equal parts) has a 50-50 chance to occur, the chance of winning the game (i.e., getting "black" on each spinner) should be also 50-50.

**Answer to the Second and Third Supporting Questions**

I answer the second and third supporting questions concurrently on the basis of the teaching experiment results (Chapter 6). First, students' informal language and strategies to reason about distributions through their qualitative and quantitative descriptions in Tasks 1-4 served as a useful starting points in specifying the conceptual trajectories. Students often reasoned about the distributions in words and drawings by articulating their attention to the various aspects of distribution, such as groups and chunks (e.g., “together” and “piled up”), spread-out-ness (e.g., “spread out,” “apart,” and “scatter out”), and shape (symmetry and skewness; e.g., “around the dot” and “pushed off that way”). For example, students showed the expected results of chips dropped from a higher position as a middle chunk for most of the chips with a wider circle around the dot (under the tube) to display the larger spread and symmetry around the middle dot. When students paid attention to different group patterns and attempted to compare them in Task 1, they usually discussed the higher and lower density areas (e.g., “close together,” “in big chunks,” and “packed together”). Further, one strategy used by a student (Josh) to quantify those groups was to divide the distribution in the picture into about equal-sized regions and compare the crowdedness in each one. The significance of this strategy was to estimate the number of buffalos in each same-size area by comparing the densities. Moreover, the variations in Task 2 (e.g., the height at which the chips were dropped and the number of chips) guided students to experiment with the chips and to develop conjectures about the distributions, particularly about the middle chunk and the spread-out-ness. These explorations were later used by the students in assigning scores for the chips landing closer or further from the target point based on the ideas of middle chunk and spread-out-ness.

Second, students’ development of conjectures and their testing of those conjectures naturally continued when they were asked to predict the outcomes, conduct experiments, and interpret the results of equiprobable events by dropping marbles in the split-box in Task 6 and flipping a coin in Task 9. More specifically, students investigated whether a marble dropped from one side or another might go to the opposite side at the bottom of the split-box in Task 6. After several trials, they developed the “opposite sides" conjecture for single marble dropping. That is, if a marble is released from the left slant, it will go to the right slot. They revised their "opposite sides" conjecture for the single marble (i.e., if all the marbles are released from the left slant, more of the marbles will go to the opposite slot) in order to predict how much more often marbles would go to the one side when they started dropping all the marbles together. In doing so, they conducted several repeated investigations with 10 marbles, such as dropping 6 marbles from the left slant and 4 marbles from the right slant, all marbles from the left slant or right slant, and 5 marbles from each slant. From these investigations (by dropping all marbles in the middle or evenly from each slant at the same time), the "middle" conjecture was developed and generalized to predict "even" or "close-to-even" results with 50 and 100 marbles in the split-box. Moreover, the "opposite sides" conjecture and the "middle" conjecture for the single marble were often used for making predictions in Task 7, a simulation of a multi-level split-box. In addition, the conception of uncertainty in the results of dropping all the marbles in the middle was considered in the uncertainty of coin flips. With that task, the "middle" conjecture was generalized to predict the numbers of Heads and Tails as "even" or "close-to-even" with the idea of "split in the middle" in the case of many coin-flips, such as 10, 50, 100, 200, and 1000. Students using this conjecture also made their predictions "even" or "close-to-even" for the spinner task (Task 10).

Third, students' reasoning about the binomial distributions of the rabbit hops and the balls in the Galton box (Tasks 11 and 13) entailed both qualitative and quantitative descriptions which eventually led to a combination of empirical and theoretical probabilities. The distributions were usually discussed in relation to the most likely and the least likely outcomes with some variations in these tasks. Students' initial ways to talk about the outcomes included the qualitative descriptions of the distributions from the simulations. These qualitative descriptions referred to middle clump, spread, and shape (symmetry and skewness), which were also often used when students reasoned about distributions in the earlier tasks. First, students needed to construct the idea of possible/impossible outcomes. Then, their initial predictions about the distribution of rabbits indicated a very limited view of possible results after five hops because they expected simply an equal number of rabbits on each side of 0 (the middle starting point) based on the equally likely hops to right and left. They began to focus on the other aspects of distributions after they conducted several simulations in the computer environment. Then, they further thought about the symmetry on the specific outcomes. For example, some students expected about an equal number of rabbits on the opposite positions, such as on -1 and +1, or -3 and +3, or -5 and +5. After several trials, those students seemed to be convinced that the rabbits were most likely to end up around -/+1 area and then -/+3 area, and least likely on -/+5, based on their revised predictions, such as more on "1s," less on "3s," and a few on "5s." In addition, students made generalizations about the location of the middle clump when the number of hops or rows of pegs was increased. Most of them thought that the range for middle clump would be a bit wider, such as "-/+1" for five hops vs. "-/+4, -/+2, and 0" for 10 hops and "$2^{nd}$-$3^{rd}$ columns" for 5 rows of pegs vs. "$4^{th}$-$7^{th}$ columns" for 10 rows of pegs. Note that the students' idea of the range of the middle clump can be possibly considered as the informal notion of the area within one standard deviation of the mean (including 68% of the data) through a normal approximation to the binomial distribution of rabbit hops.

Then, their predictions for the range seem to be reasonable because if one were to simulate 10,000 rabbits hopping 5 times and 10 times, the standard deviation would be approximately 2 and 3, respectively, which are close to the students' predictions. Furthermore, students thought that if the number of rabbits or balls were increased, they would "scatter around or distribute more" meaning with similar shape but more in them (sample space). When students investigated the skewed distributions in both tasks with the computer simulation, they expected that the middle clump would shift to the right (e.g., "6") or left (e.g., "-6") with respect to the middle (e.g., "0") if the chance of hopping or bouncing right changed to 75% and 25%.

Most students (Alicia, Emily, Caleb, and Josh) also often used paths as inscriptions to initially reason about the possible outcomes. Then, some students, especially Caleb and Josh, reasoned with paths to figure out the number of ways to get to the most likely and the least likely outcomes. For example, students began referring to paths (verbally) to talk about possible ways to arrive at one of the bottom lanes in the multi-level split-box game (Task 7), such as "[Maya] It goes there [left], then it goes there [left], again it goes there [left], then it again goes there [left], and again it goes there [left and lane 1]." Then, some students (Josh and Alex) noticed different patterns (on the counters) to get to the same lane when discussing where the counters arrived mostly at the end of the game. This critical observation led students to consider the number of ways to get a varied sequence of "left" and "right." For instance, Emily and Alicia indicated that there were many ways to get to lane 3 based on the game results (the number of counters in the lane and different patterns of three left-turns and two right-turns). However, only Josh explicitly described the patterns in terms of a combination of left turns and right turns, such as “[Josh] These two, right here, are almost the same because on one of them it is RRLLL and on the other one it is LRRLL. So, both of them have three lefts and two rights.” Therefore, Josh made the first linkage between permutations (different sequence of R and L) and combinations (number of R and number of L in these sequences) in the multi-level split-box game by reasoning with paths. Furthermore, all of the students began to develop the notion of sample space in the *Hopping Rabbits* task by generating possible paths of five hops on the number line, such as if there were an odd (or even) number of hops, the rabbits would land on the odd (or even) numbers. Moreover, students had intuitive ideas about the number of ways to get to each outcome when they talked about the most likely and the least likely ones in terms of “easier to get” and “harder to get” paths, respectively. “Easier to get” paths were usually considered to be the ones with “even” or “close-to-even” number of Heads and Tails (or Left and Right), consistent with their previous beliefs about the outcomes of equiprobable events, such as the split-box, coin flipping, and the spinner task. Students also were encouraged to develop an understanding of combinations and permutations as a way to justify the number of ways to get to each outcome. For example, they were asked to generate all possible ways to land on the possible outcomes on the number line, e.g., -/+1, -/+3, and -/+5, for rabbits after five hops. When they completed their list, some (i.e., Caleb, Emily, and Josh) began to reason that the rabbits were more likely to land on -/+1 because they had more possible ways (ten for each outcome) to happen.

Students seemed to have empirical intuitions which could be developed into a system of quantitative relationships based on the relative frequencies. Once students developed the notion of sample space and conducted several computer simulations repeatedly for different numbers of rabbits, they tended to quantify the most likely and the least likely outcomes based on either the previous simulation results or the number of ways to get to each outcome. These two approaches entailed the intuitive conceptions of (1) empirical probability and (2) theoretical probability. The development of the quantification of likelihood of outcomes began with the number of rabbits on the right side and on the left side of the middle starting point, such as “6 on the right and 4 on the left.” As students did more and more computer simulations, they tended to combine the qualitative and quantitative aspects of distributions by focusing on the symmetry of the outcomes on each side of 0. Then, they often reasoned about the numbers across possible outcomes, such as “60 for 1s, 30 for 3s, and 10 for 5s,” with symmetry (i.e., “even” or “close-to-even” on – 1 and + 1). When these numbers for quantifying the likelihoods of the outcomes were revised according to the previous simulation data, their ratios to the total number (empirical probabilities), if calculated, tended to approach to the theoretical probabilities. After students generated all possible outcomes (e.g., the permutations of Heads and Teals in five coin-tosses) as an attempt to find the number of ways to get to each outcome (e.g., the number of permutations for each combination of Heads and Tails in five coin-tosses), they reasoned about the likelihood of each outcome (1s, 3s, and 5s) with the number of ways. While most students were able to compare the probabilities (from “biggest chance” to “smallest chance”) qualitatively, only one student (Josh) who developed the ideas of permutations and combinations calculated the theoretical probabilities, such as the compound probabilities P(1s)=20/32, P(3s)=10/32, and P(5s)=2/32. Hence, the development of quantifying the theoretical probabilities (e.g., “the number of ways to get each outcome” divided by “the number of all possible combinations") required students to both construct the combinatoric operations (combinations and permutations) and establish the relationships of the individual cases with the whole distribution (Piaget & Inhelder, 1975).

**Answer to the Fourth Supporting Question**

The post-interview findings addressed the fourth supporting question. The results were discussed in detail in Chapter 7. Thus, here I present the major findings in relation to learned ideas over the course of the teaching experiment. Those included students' reasoning about equiprobable and non-equiprobable situations, ideas about combinations and permutations, conceptions of probability, prediction about a sample in proportion to the population, and understanding of probability of compound events.

The students' responses in the post-interviews demonstrated that students were more attentive to the aspects of equiprobable and non-equiprobable routes in the five different channel systems. Particularly, their justifications seemed to be straightforward for the channel systems with two and four equiprobable routes, i.e., "it can go either way." However, students needed to simplify the situation in the channel system with eight equiprobable routes and focus on the routes to exits 1 and 2 in order to justify the equiprobability, like Emily did. When explaining the non-equiprobable routes, students mostly relied on visual justification with an intuitive idea of the multiplication principle of probability, such as "this is easier [to go to exit 1] because if it goes down here [right], it could go to either one of these [exits 2 and 3]." Another approach (probabilistic or mathematical) was to calculate the probabilities by using the multiplication of probabilities, like Josh's solution "50-25-25 chance" (in figure C, see Appendix A, interview task 1), to verify the non-equiprobable routes. For the other non-equiprobable situation (figure E), students needed to reason with either the "number of ways" visually, i.e., "[Emily] There are two ways to get to 2 because say the little marble went down here [left] and then it went down here, then it could go in to 1 or 2. But say the little marble went down here, it's going to go in to 2 or 3 which means there is two ways to go on to the 2," or the addition principle of probabilities, e.g., "[Josh] If the one right here, you know the ball could go to 2 or 1. But let's just say one over here, then it could go to 2 or 3. So, this [exit 1] is more like of a 25% chance, this [exit 2] is 50, and this one [exit 3] is 25." Finally, students tended to use probabilistic reasoning more often in the post-interviews when they referred back to the particular tasks they did in the teaching experiment to explain their thinking.

With regard to combinations and permutations, most of the students could generate them in a systematic way. For example, all students provided written or verbal explanations that confirmed a recognizable strategy for finding all possible combinations of three flavors of ice cream served in three different types of container. Moreover, one student's (Josh) justification of the number of all possible combinations involved a symbolic representation, such as the multiplication rule (a basic counting principle). Four of these students also were able to generate all possible combinations of two elements whereas the other two could only think of mixed pairs, such as blue-yellow. Furthermore, one student (Alicia) explicitly mentioned the importance of order, e.g., "$1^{st}$ and $2^{nd}$," in generating the possible pairs, such as blue-yellow and yellow-blue. Another student (Josh) justified his list of possible pairs by referring to the tasks in which they arranged five blue and red bears in a row in thirty-two different ways and listed all possible paths of five hops for rabbits by switching the order.

All students again were able to reason about the given frequencies to talk about the probability. Furthermore, in addition to the two students (Emily and Josh) prior to the teaching experiment, two other students (Alicia and Caleb) constructed the relationship between the part and the whole. Therefore, they could represent probability as a ratio of the number of favorable cases to the number of all possible outcomes.

In the *Gumballs* task, majority of the students took into account the distribution of gumballs and thus predicted the number of red gumballs in the sample in proportion to the population. Even though they seemed to appreciate that there were 50% red gumballs, most of them did not explicitly indicate a reason in terms of the proportion of the red gumballs in the mixture.

Prior to the teaching experiment, none of the students could reason about the chances involved in a compound event, i.e. two spinners with two equal parts. At the end, only one student (Josh) was able to reason with the sample space in order to justify that the chance of winning the game was not "50-50." Therefore, he calculated the theoretical probability of winning the game as he did in the various chance events during the teaching experiment, such as the sum of two dice and the binomial rabbit hops.

## Conceptual Corridor

In line with the goal of design experiments stated by Confrey (2006), the purpose of this study was to model the conceptual corridor involving possible conceptual trajectories taken by students based on their conceptions of probability and reasoning about distributions in chance events, using a sequence designed tasks. In constructing that corridor, the design and sequence of tasks discussed in Chapter 4 constituted the borders of the corridor by forming a set of constraints. Hence, certain types of distributions and probabilistic conceptions were approached through this conceptual corridor. For example, the kinds of distributions included the pictures of natural distributions, the centered distributions of chips generated on a three-dimensional setting, the uniform distribution of marbles, the triangular distribution of the sum of two dice, and the binomial distributions of random rabbit hops and the balls in the Galton box. The reasoning about these distributions as aggregates also entailed certain features of distributions, such as middle, clump, spread, and shape (symmetry and skewness). Furthermore, the probabilistic conceptions focused on in these tasks involved equiprobability, sample space, combinations and permutations, the law of large numbers, empirical probability, and theoretical probability.

The students' prior knowledge about particular probability concepts and their reasoning before entering this conceptual corridor were addressed above by answering the first supporting question. Building on this prior knowledge and my conjectures (see Chapter 4), students' responses were initially anticipated and then documented during the teaching episodes (discussed in Chapter 6). In addition, the answer to the second and third supporting questions (discussed above) addressed the student interactions across tasks throughout the teaching experiment. Considering these resources, I specified the landmark conceptions, the obstacles, and the opportunities in the conceptual corridor. In the following subsections, I discuss each component that comprised the corridor.

**Landmark Conceptions:** Students' qualitative and quantitative reasoning about distributions in chance events, and their intuitive conceptions of empirical and theoretical probabilities generated a set of *landmarks* in this corridor. More specifically, the qualitative reasoning about distributions involved the conceptions of groups and chunks, middle clump, spread-out-ness, density, symmetry and skewness in shapes, and "easy to get/hard to get" outcomes. Furthermore, students' quantitative reasoning arose from these qualitative descriptions of distributions when they focused on different group patterns and compared them to each other. Initially, students began reasoning about distributions qualitatively when they were attentive to the various features of distributions shown in the pictures (Task 1: *Distributions in Different Settings*). For example, they often indicated where the buffalos were "together" or "apart" in the picture. Then, their focus on different patterns (e.g., "packed together" and "scatter out") in the buffalo picture led them to consider density as a feature to distinguish some patterns from the others. Eventually, the density notion was used by Josh in estimating the number of buffalos in a field by comparing the crowdedness in the equal-sized areas.

Once students began to reason about the binomial distribution of random rabbit hops, their reasoning about the most likely and least likely outcomes were initially based on the "easy to get" and "hard to get" paths. In transition to the idea of paths, students needed to develop an understanding of equiprobable events (or the notion of 50-50) in the earlier tasks that involved dropping marbles in the split-box and flipping a coin. For example, since the rabbits had equal chances for hopping right and left, each path of left and right hops, such as LRRLL, LLRRR, LRLRL, and so on, was equally likely to happen. However, the final outcome (the position after five hops on the number line) would yield different chances for different combinations of left and right hops. For instance, the rabbits would have more chance to land on either -1 or +1 more than to land on either -3 or +3 since they could hop to -/+ 1 in more different ways than to -/+3. To support this idea through a trajectory from empirical reasoning to theoretical reasoning, the students were asked to conduct various computer simulations of random rabbit hops and to explain the results in terms of the most likely and least likely outcomes (the final positions). Their interpretations of these empirical results from the computer simulations began to develop from their reasoning about symmetry, middle clump, and spread in the distributions into quantifying those outcomes based on the frequency of data. Students' strategy of revising their predicted quantities on the basis of the previous results entailed an intuitive idea of empirical probabilities. For example, students tended to make their predictions first based on the proportional model from the previous results when the number of rabbits increased and then modified them as they ran more simulations with the same number of rabbits.

While thinking about the likelihood of outcomes through simulations, students needed to develop an understanding of permutations and combinations of left and right hops in order to move beyond just an empirical generalization. Since students already had ways to talk about the most likely and the least likely outcomes in terms of "easy to get" and "hard to get" paths, they began to list of all possible paths for those outcomes, such as -/+1, -/+3, and -/+5. Furthermore, constructing the ideas of combinatoric operations (e.g., permutations and combinations) provided students a way to reason with the number of ways to get the outcomes for comparing the likelihood of each outcome (e.g., "1s" were more likely because they had "more options which means more available”). Then, with the foundation of the relationship between the single cases in the sample space and the whole distribution, the notion of theoretical probability was constructed in the form of “the number of ways to get each outcome” divided by “the number of all possible paths.” In this study, there was only one student (Josh) who got the idea of theoretical probability because he initially had an understanding of the relationship between the part and the whole in quantifying probability (discussed in the pre-interview results, Chapter 5). Then, during the teaching experiment, he constructed the relationship between different combinations and all equally likely outcomes (or paths). Furthermore, Emily came closest to Josh’s thinking when she ordered the chances of landing on “1s, 3s, and 5s” from biggest to smallest, such as $P(1s) > P(3s) > P(5s)$, based on the list of all possible paths for those outcomes, but with no quantification. Like Josh, Emily also initially had a conception of part-whole relationship to calculate the probability of a single event (Chapter 5), but she was not able to form the same relationship using the number of paths in each different combination of left and right hops and the number of all equally likely outcomes for five random hops.

**Obstacles:** In modeling the conceptual corridor for this teaching experiment, the *obstacles* were important to note since the conceptual trajectories taken by students were also determined by these realizations. In the earlier tasks in the sequence, students mostly relied on deterministic and causal reasoning about distributions. For example, they interpreted the distributions of marbles in the split-box activity as “deterministic physical models[2]” (Metz, 1998, p. 304) when investigating a mechanism to find a systematic way to predict the outcomes by dropping the marbles from the opposite slant.

[2] This phrase was used to describe children’s predictions based on the analysis of the physics of the apparatus and the marble movement in the apparatus when the physical model supported precise predictions (Metz, 1998).

However, they came to understand that if all marbles were dropped in the middle or equally from each slant at the same time, anything could happen because the marbles might go either way when they hit the middle divider. Then, by watching the movement of the marbles in the split-box, they thought that the marbles would knock each other out and go to different sides almost evenly. Moreover, students tended to use deterministic reasoning when they conducted real experiments with picking gumballs from the gumball machine. For instance, when a gumball was picked and was not the color they predicted based on the proportion of colors in the mixture, one student argued that the gumballs might have been hidden at the bottom. I interpreted this reasoning as the absence of the conception of variation in a small sample because in a larger sample one would be more likely to get the color of gumball with the highest proportion in the mixture.

Students also seemed to lack an understanding of the law of large numbers. More specifically, students believed that the chance based differences in the number of marbles in the left and right slots in the split-box would get larger proportionally as the number of marbles increased. For example, some of the responses were “7 on the right and 3 on the left” for 10 marbles and “700 on the right and 300 on the left” for 1000 marbles. It is usually true that when the sample size increases, the absolute difference between the numbers in the right and left slots gets larger, but the percentage of the marbles in the right slot settles down to 50%. However, some students often did not recognize the relative differences; instead they used the proportional model. A different example of this kind of reasoning was seen in students’ expectations about “10-difference” between the numbers of heads and tails, such as “40T-60H,” “110T-90H,” and “510T-490H,” no matter what the sample size was.

Students' misconceptions, personal beliefs, and judgmental heuristics furthermore led them to employ erroneous conceptions and strategies in judging the likelihood of uncertain events. In rolling two dice, one student believed that getting a larger sum was more likely to happen because there was a less chance of getting low numbers in rolling one die. Her belief was persistent even when all possible ways to get each outcome were listed to find out the number of ways for each sum. Her reasoning could be interpreted as the availability heuristic, a bias due to the person's own limited experience and personal view (Tversky & Kahneman, 1973).

Students also often reasoned that getting the same side of a coin in 5 tosses would not happen frequently, such as "You don't usually land on HHHHH or TTTTT. You usually land on like TTTHH." One interpretation of this kind of reasoning might be the representativeness heuristic on the basis of how well it was representative of the 50-50 chance of Heads and Tails to occur in the population (Kahneman & Tversky, 1972). In other words, paths, like TTTHH, are considered more likely to happen than those, like HHHHH or TTTTT, because the former seem to have an equal number of Heads and Tails and, thus, be more representative of the 50-50 distribution of Heads and Tails in flipping a coin. The readers also should note that the language used in their responses, such as "land on *like*" and "get *like*," did not suggest a certain path or sequence of Heads and Tails in terms of a pattern or randomness. Rather, it can be argued that the use of "like" prior to any path refers to all possible paths with "even" or "close-to-even" proportion of Heads and Tails in the sequence. In contrast to the conception of "even" or "close-to-even" results in equiprobable events, students' initial way of reasoning with "anything could happen" was indicative of the outcome approach (Konold, 1991). The change in students' thinking might be a result of their hands-on experiences during the teaching experiment supported by the tasks designed to investigate equiprobable events, such as the split-box and marbles, through student-generated investigations and their analyses of outcomes.

**Opportunities:** Over the course of the conceptual corridor, certain *opportunities* came from student interactions of the particular group or affordances of a particular tool. Those involved student-generated conjectures and inscriptions, building common meanings for language, and support for mediational actions. Throughout the tasks, students were encouraged to develop and test their conjectures when asked to predict-experiment-interpret the results. For example, they were initially guided to conduct several investigations to make conjectures about the effect of the height at which the chips were dropped. Then, their own investigations with the marbles in the split-box were used to test and develop the "middle" conjecture which referred to "even" or "close-to-even" dispersion of the marbles when they were dropped from the middle or evenly from each slant at the same time.

Moreover, students tended to generate inscriptions of paths in order to reason about the possible ways to get to an outcome in a random binomial situation. These inscriptions of paths helped them notice the possible and impossible outcomes of five random rabbit-hops. Then, they developed a generalizable conjecture that if there were an odd (or even) number of hops, the rabbits would land on the odd (or even) numbers. The paths as inscriptions later became a tool for reasoning about the number of ways to get to each outcome for five and ten hops. More specially, some students, namely Caleb and Josh, considered the number of ways to get to an outcome when making their predictions about the most likely places for rabbits to land on after 10 hops. To do so, they generated inscriptions of paths to keep track of ten hops.

Within the socio-cultural setting of the study, building a common language was another way to discuss *opportunities* in the conceptual corridor. After having been exposed to informal discussion about how things were distributed in various settings, such as buffalos in a field, leaves under the tree, and the chips dropped through the tube, students were introduced to the term "distribution" as used in mathematics in the second teaching episode. After that, I consistently used the term "distribution" with the marbles, rabbit hops, and balls in the Galton box. Their language often referred to certain characteristics of distribution, such as the middle, spread, and shape, while only one student (Emily) used the action verb, *to distribute*, in the discussion of distributions of balls in the Galton box.

Furthermore, particular mediational actions (Wertsch, 1985) were supported in this conceptual corridor. For instance, in the fifth teaching episode, students first noticed different patterns to get to the same place as an outcome when discussing the results of the multi-level split-box game. Their naïve understandings of combinations and permutations were then supported by the task in the sixth teaching episode in which students were given five blue and five red bears and asked to generate as many different ways as they could to arrange five bears in a row. This activity helped students develop a systematic way to generate all permutations. Also, one student who noted the different combinations in this task was then able to reason about the number of ways to get to each outcome in the *Hopping Rabbits* task. Further, he used his reasoning about the number of ways to discuss the most and the least likely outcomes.

The use of socio-cultural tools, such as the NetLogo computer environment for the *Hopping Rabbits* and the *Galton Box* tasks, mediated certain actions. For example, the computer simulations provided students an opportunity to modify particular elements of the binomial distributions of rabbits or balls, such as the number of hops or rows of pegs, the chance of hopping or bouncing right. Moreover, conducting larger numbers of repeated trials with the computer environment helped students develop the intuitive ideas about the empirical probabilities. When they began to quantify the likelihoods of outcomes for rabbit hops, they often revised their predictions for the number of rabbits in the possible outcomes based on the repeated trials.

So far, the conceptual corridor has been modeled starting from the students' prior knowledge and through the *landmark conceptions*, *obstacles*, and *opportunities* that formed a possible trajectory for the students. Finally, the findings regarding the fourth supporting question, discussed above, inform readers about the resources students brought into their reasoning in the interview questions after engaging in reasoning about distributions in chance situations through this conceptual space.

To summarize, the trajectory discussed in this conceptual corridor showed how students' qualitative reasoning about distributions evolved into a quantitative reasoning throughout their engagements in the sequence of tasks. Previous studies discussed in Chapter 2 established certain pieces of this trajectory. For example, Cobb and his colleagues (Cobb, 1999) and Lehrer and Shauble (2000) documented how middle school and elementary school students reasoned about distributions by exploring the qualitative characteristics of distributions in the context of data analysis. Similarly, the students in this study used informal language, such as chunks, spread out, and bunched up, to describe the aggregate, spread, density, and shape of distributions in chance events. Along the conceptual corridor, some of the students' responses about outcomes of a chance event indicated particular misconceptions, such as representativeness, availability heuristics, and outcome approach, documented by other researchers on specific tasks (e.g., Kahneman & Tversky, 1972; Konold, 1991; Tversky & Kahneman, 1973). Moreover, the development of students' formal understandings of empirical and theoretical probabilities in this study was supported by the work of Piaget and Inhelder (1975). This study also added a new dimension to our knowledge of children's understandings of binomial probability distributions, particularly through students' reasoning about paths which were sometimes shown by inscriptions, i.e., random rabbit hops, and sometimes visually apparent, i.e., Galton Box model.

## Limitations

This study had some limitations. The sample of the study was a convenience sample. The participants were not selected randomly. I recruited students through their classroom teacher who selected the volunteered students based on the permission of their parents for video- and audio-taping. Then, I worked with the students in two groups of three during the small-group teaching experiment study. Therefore, its small convenience sample may warrant a limitation in generalizing the findings of this study to a broader population of students. Moreover, a particular set of tasks and a specific sequence of these tasks were used in this study. Then, I presented a way to look at students' development of understanding of probability concepts and probabilistic reasoning. Thus, the conceptual corridor does not apply to an absolute developmental path. Rather, it refers to a plausible trajectory towards a more sophisticated understanding of probability concepts.

Even though the data for this study came from a variety of sources, I was the primary researcher for collecting and analyzing the data. Hence, my background and teaching experiences might have a potential bias in the process of data collection and qualitative analyses. In order to minimize the possibility of such bias, I kept a notebook on a daily basis to record my thoughts, reflections, insights, and decisions made after each teaching episode during the teaching experiment. Moreover, the triangulation of data from multiple sources was used to confirm emerging themes in the analyses.

## Implications

One implication of this study entails how the findings of this study contribute to the body of research on children's understandings about chance and data. There have been numerous research studies on students' conceptions of these topics at different age groups (as discussed in Chapter 3). Much of the recent research on students' conceptions of data involved how students reason about distributions in the context of data analysis. Moreover, research on students' conceptions of chance investigated different kinds of reasoning under uncertainty, the development of the probability concept in children, and the models of children's probabilistic reasoning. Recently, there has been an increased interest in linking the discussions of data and chance, both in research and teaching, among the statistics education community. For example, the Model Chance project which started two years ago aimed at building tighter links between the probability and data strands in the mathematics curriculum prior to the high school (Konold, 2004).

Thus, the research team currently focused on developing a probability simulation tool integrated TinkerPlots (Konold & Miller, 2004), data analysis software, and curriculum materials to help middle school students learn about probability through modeling chance events.

I believe this study contributes to the current state of research by offering a possible new direction to ways of connecting chance and data through reasoning about distributions in chance events. Furthermore, the major contribution of this design study was to articulate possible elements of a conceptual corridor that entailed likely conceptual trajectories taken by students based on their conceptions of probability and reasoning about distributions in chance events through a sequence of tasks. Hence, it documented the landmark conceptions and obstacles students have and opportunities to support students' development of probabilistic concepts. Moreover, the study has shown that fourth graders began to reason about probability distributions, such as the binomial distributions of rabbit hops and the balls in the Galton box, with the development of probabilistic conceptions, such as permutations and combinations, and sample space.

Another implication of the study involves how the resources the students bring into understandings of probabilistic concepts and distribution can inform the practice. This study provided an analysis of a conceptual corridor in which fourth-grade students began to reason about distributions with the intuitive ideas about probability in a sequence of tasks. Thus, it is important to link discussions of data and chance starting from the elementary school probability and data analysis strands in the mathematics curriculum. Moreover, the study has shown that students could be encouraged to develop and test conjectures relevant to the uncertainty of the event through "predict-experiment-interpret results" with the simulations. Also, the use of appropriate technological tools, such as the NetLogo modeling environment, should be integrated into the discussions of data and chance.

## Future Research

In this design study, the conceptual corridor defined a possible trajectory in which fourth-grade students both developed probabilistic ideas and reasoned about distributions. As mentioned before, this corridor was constrained by the design and sequence of tasks discussed in Chapter 4. Therefore, it is possible to develop other ways to approach the ideas of chance and data with the design of another series of tasks. For example, the next level where chance and data topics are usually connected is the statistical inference. Hence, research is also needed to design and sequence tasks through which students can develop a conceptual understanding of statistical inference and to document how their informal conceptions evolve into the formal ideas of statistical inference.

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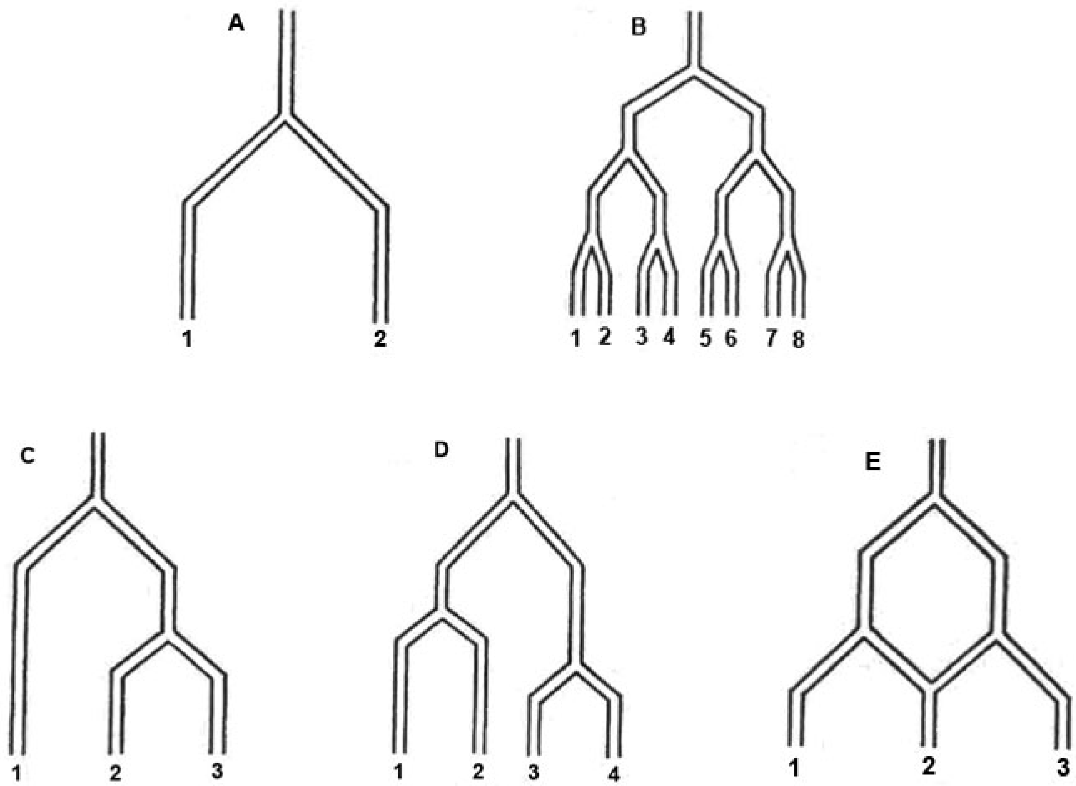
## APPENDIX A

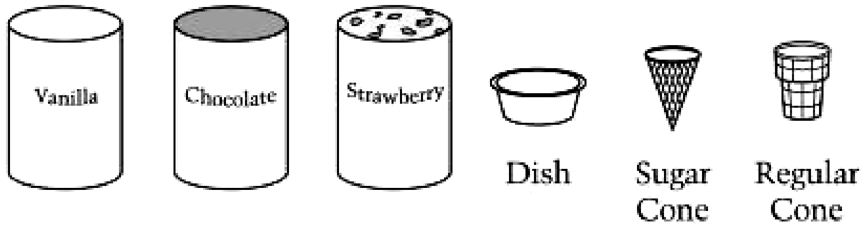
## INTERVIEW TASKS

1) This is type of a guessing game. There are five figures showing the different channels (see A, B, C, D, and E). Suppose we place a marble at the top and let it drop many times one after the other, and then it will come out at the bottom in one of these numbered exits.

Circle the figures where you think that the marble is as likely to come out of exit 2 as exit 1. Explain how you got that answer.

(Source: Fischbein, E., Pampu, I., & Minzat, I. (1967). The child's intuition of probability. *Enfance, 2*, 193-206. Reprinted in Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children, 156-174*. Dordrecht, The Netherlands: Reidel.)






2) Jan's Snack Shop has 3 flavors of ice cream: vanilla, chocolate, and strawberry. The ice cream can be served in a dish, a sugar cone, or a regular cone.

- There are 9 people who choose 1 dip of ice cream in a dish, or in a sugar cone, or in a regular cone, and all of their choices are different. List or show the 9 different choices.
- Could another person have a choice that is different from one of these 9 choices? Why or why not?

(NAEP, 2003-4M10)

3) There are 3 fifth graders and 2 sixth graders on the swim team. Everyone's name is put in a hat and the captain is chosen by picking one name. What are the chances that the captain will be a fifth grader?

A) 1 out of 5
B) 1 out of 3
C) 3 out of 5
D) 2 out of 3

Explain how you got that answer.
(NAEP, 1996-4M9)

| *Stickers* | *Number* |
|---|---|
| Red | \|\|\| |
| Blue | \|\|\|\| |
| Yellow | \|\| |
| Green | ~~\|\|\|\|~~ \|\| |

4) The 16 stickers listed above are placed in a box. If one sticker is drawn from the box, which color is it most likely to be?

A) Red
B) Blue
C) Yellow
D) Green

Explain how you got that answer.

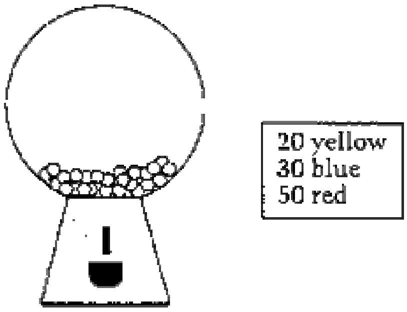
(NAEP, 1990-4M7)

5) Steve was asked to pick two marbles from a bag of yellow marbles and blue marbles. One possible result was one yellow marble first and one blue marble second. He wrote this result in the table below. List all of the other possible results that Steve could get.

| First Marble | Second Marble |
|---|---|
| y | b |
| | |
| | |
| | |

y stands for one yellow marble.
b stands for one blue marble.

(NAEP, 1992-4M7)



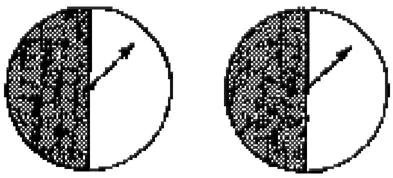

6) The gum ball machine has 100 gum balls; 20 are yellow, 30 are blue, and 50 are red. The gum balls are well mixed inside the machine.

Jenny gets 10 gum balls from this machine.

What is your best prediction of the number that will be red?

Answer:____________________ gum balls

Explain why you chose this number.
(NAEP, 1996-4M12)



7) The two fair spinners shown above are part of a carnival game. A player wins a prize only when both arrows land on black after each spinner has been spun once. James thinks he has a 50-50 chance of winning. Do you agree? Justify your answer.
(NAEP, 1996 -12M12)

## APPENDIX B

## THE RUBRIC FOR SCORING THE PARTICIPANT'S RESPONSES IN THE PRE-INTERVIEWS

| **Tasks** | **Rubric** |
|---|---|
| **Task 1:** | (3) Correct figures w/ probabilistic reasoning |
| **Channels** | (2) Some correct figures w/ probabilistic reasoning |
| | (1) Some correct figures w/ non-probabilistic reasoning |
| | (0) None |
| **Task 2:** | (2) All possible combinations |
| **Ice-Cream** | (1) Some possible combinations |
| | (0) None |
| **Task 3:** | (2) Correct answer w/ part-whole relationship |
| **Swim Team** | (1) Correct answer w/ no part-whole relationship |
| | (0) Incorrect answer |
| **Task 4:** | (2) Correct answer w/ an explanation with frequencies |
| **Stickers** | (1) Correct answer w/ no satisfactory explanation |
| | (0) Incorrect answer |
| **Task 5:** | (3) All possible permutations |
| **Marbles** | (2) Only two possible permutations |
| | (1) Only one possible permutation |
| | (0) None |
| **Task 6:** | (2) "4-6" w/ proportional reasoning or w/ a satisfactory explanation |
| **Gumballs** | (0) Incorrect answer |
| **Task 7:** | (2) "Don't agree" w/ probabilistic reasoning (sample space or |
| **Spinners** | multiplication rule) |
| | (0) Incorrect answer |

# APPENDIX C

# PILOT STUDY TASKS

## Distributions in Different Settings

Students are shown fourteen different (digital) pictures of distributions in various settings one at a time. Those included bees on a honeycomb, a kid walking among pigeons in a Square, a buffalo herd, a sheep herd, penguins, a group of swimming fish, wild flowers in a plateau, foliage in a forest, leaves under a tree in the fall, chocolate chips cookies, airline routes from one location to other cities in the U.S.

Questions:

1. What do you notice in the picture?
2. Do you see any pattern?
3. Where do you see more of them? Less? Why? How do you know?

## Dropping Chips Experiment

<u>Materials</u>: Plain poster sheets, red/blue/white chips, two tubes, measuring tapes, markers with different colors.

<u>Demo</u>: Place the tube 15 inches above the center of the sheet on the floor. Drop all 20 chips at the same time through the tube at that height. Then each pair conducts their own experiments and we discuss the results all together.

<u>Experiment 1</u>: (Height=15 inches)

Trial 1: Predict/Experiment/Interpret (Describe the pattern of the chips on the sheet)

Question: If you did this experiment again, would it be the same as this one? How? Why?

Trial 2: Predict/Experiment/Interpret (Compare the two trials. Explain and interpret the similarities and differences between the two trials. Leave the chips on the floor as they are.)

Experiment 2: (Height=30 inches)

Trial 3: Predict/Experiment/Interpret (Make a guess about the distribution of chips. Drop another set of twenty chips. Compare the results to the last two distributions from Trial 2.)

Questions: Looking at all three distributions, what can you tell about the patterns? Where are the most of them? How does the height affect how close they can be under the tube?

Design Your Own Game: Suppose you are making a game and want to give different points for landing near or far from a target. Conduct experiments to determine: How does the height of the tube affect how close chips can come to the target? Question: Explain why you designed the game this way? Why did you assign these numbers to each particular region? (Play each other's game.)

**Split-box**

This task adapted from the study of Piaget and Inhelder (1975) will be used to investigate students' understanding of distributions that are generated with the notion of chance inherent in the physical objects/apparatus. More specifically, students will experiment with an inclined box with a centered funnel-like opening on the upper part to drop the marbles. The split-box has a partition dividing the lower side into two same-size slots The purpose of this task is to examine students' interpretations of the results and how they use the preceding observations.

Experiment With Replacement: Each student will drop a marble. Before ach trial, they will make predictions about whether the marble will go to right or left slot. (The total of four marble-drops)

Experiment Without Replacement: Student will drop 10, 50, and 100 marbles. They will make predictions before each trial and then explain the resulting distribution of marbles after each trial. They will also compare the expected and the resulting outcomes.

**Flipping a Coin**

Questions:

1. When do we toss a coin? For what purposes?
2. What are the possible outcomes when you toss a coin?
3. Before we start tossing a coin, can you predict whether you will get Heads or Tails in the first trial? What would you expect in the second trial? And in the third? Fourth? Fifth?
4. How many Heads would you expect if you tossed a coin 5/10/50 times?
5. After 5 and 10 coin-tosses, record your results on the paper. How many Heads did you get? Are you surprised? Why or why not?
6. Compare the expected and the resulting outcomes.

**Hopping Rabbits**

Problem: Suppose there are a number of rabbits on a land where each rabbit can choose to hop only right or left. For each hop, rabbits are as just likely to hop right as left. We want to know where a rabbit is likely to be after five hops.

Questions:

1. What do you think if there is a bunch of rabbits, where they would be most/least likely after five hops? Remember they can only hop right or left.
2. Show them the number n\line and ask the same question.
3. How about we toss a coin to decide where the rabbits will hop next to determine where the rabbits will be after five hops by tossing the coin five times?

Experiment: Working with your partner, toss a coin five times to find where the rabbit will be after five hops. Generate results for 25 or more trials. As one person is flipping the coin, the other person should keep the record of each hop and mark where the rabbits end up on the number line after five hops.

Whole-class Discussion: First, discuss the results from each experiment and then compare the both.

Questions:

1. What do you notice? Do you see any pattern?
2. According to these results, where will be more of them? And fewer?
3. Compare the two resulting distribution. According to which experiment are the rabbits more likely to be on -1? (Compare fractions or percents)
4. What do you notice about the possible outcomes?

5. What happens if we combine the results from the two experiments? What do you notice? Do you see any pattern?
6. Do you expect that rabbits are equally likely to be on -1 and 1 (or either side of 0) if we continue to collect more data?
7. Do you see any places where they are all together/jumbled up/grouped? Do you see any places where they look scattered/separate? Do you see any pattern when you look at this chart? (Discuss the most likely and least likely outcomes, and statistical features of the distribution, such as center, spread, and shape.)
8. Can you tell me what the possible outcomes are after five hops?
9. With your partner, generate all possible ways a rabbit could get to that position when you toss a coin five times.
10. How many ways are there to get 3? -1? 5? -2?
11. Where is a rabbit more likely to be after five hops simulated by tossing a coin five times?
12. How likely is a rabbit to be on 5? -5? 1? -1? 3? -3?

## APPENDIX D

## (REVISED) TASKS USED IN THE TEACHING EXPERIMENT STUDY

### Tasks Used in Teaching Episode 1

**Task 1: Distributions in Different Settings**

“Buffalo Herd” Picture:

General questions: What do you notice in this picture? What else do you see? What can you tell about patterns or arrangements of buffalos in the picture?

Why question: Can you explain what you mean by …. [students’ language, such as “gathered”, “spaced out”, “jamming up”]? What do you notice about the rest? Can you explain why it happens to be this way? Does it matter how they are organized? (To see whether they distinguish causally determined and chance-based events)

Probing questions: What do you mean by “a lot of”? How would you guess the number of buffalos? Are there any places where there are more buffalos than the other places? Where do you see more of them? Where do you see less of them? Can you guess how many buffalos there are in this picture? How did you predict? Can you explain to me how you did it?

“Leaves in the Fall” Picture:

General questions: What do you notice about the leaves on the ground? What can you tell about the way leaves fell on the ground? Do you see a pattern of leaves falling off the tree on the ground?

Why question: Can you explain why the leaves fell on the ground this way? Does it matter how they are organized?

Probing questions: Where do you see more of the leaves falling off from the tree on the ground? And where do you see less of them?

"Airline Routes Map" Picture:

General questions: What do you notice about these nonstop flights on these two maps? What can you tell about the way the nonstop flights are arranged for both cities? Do you see a pattern for the destination cities?

Why question: Can you explain why the nonstop flights from these two cities are arranged this way? Does it matter how they are organized?

Probing questions: Where do you see more of the flights out of Birmingham on the map? And where do you see more of them out of Dallas on the map? (Note that we cannot tell how many flights there are from these cities to the destination cities daily looking at these maps.) Are there more flights from one of these cities? Why do you think there are more longer flights out of Birmingham than Dallas?

## Tasks Used in Teaching Episodes 2 and 3

### Task 2: Dropping Chips Experiment

Materials: Chips (20 blue, 20 red, and 20 white), two tubes, two measuring tapes, color markers, and two plain poster sheet (32x27 inches).

Demo: Explain how to use the tube, to drop the chips all together through the tube, to use the measuring tape to see how high the bottom of the tube is above the ground to drop the chips, and to mark the location of the tube on the sheet.

Predict: What do you expect when you drop these twenty blue chips through the tube at this height (15 inches) on to the middle of big white sheet (32x27 inches)? Where do you think most chips will land on the sheet? Can you mark that region on the sheet? Why do you think so? Do you expect any chips would land on outside of that region?

Experiment: [Dropping chips]

Interpret: What do you notice about the chips? What do you think about these results and your predicted results you told me before the experiment? Can you describe and explain the similarities/differences you see between the actual results and your predictions?

<u>Experiment I</u>: [Students will use the same sheet with different color marker.]

Predict: If you conduct a new experiment with 20 chips (height=15”) again, do you expect the same results (as in the previous experiment)? Discuss in your group first. Can you explain why or why not? Can you show me (or mark on the sheet) where you think most of the chips will be?

Experiment: [Dropping chips]

Interpret: What do you notice about the chips? What do you think about these results and your predicted results before the experiment? Can you describe and explain the similarities/differences you see between the actual results and your predictions? What is the same and what is different between these results and the previous ones? Can you explain why it happened that way?

<u>Experiment II</u>: [Variation in the height]

Predict: If you conduct a new experiment with 20 chips and this time double the height at which you drop the chips (height=30”), can you make a prediction about the results? Do you expect the same results (as in the previous experiment)? Can you explain why or why not? Can you show me (or mark on the sheet) where you think most of the chips will be in this experiment?

Experiment: [20 chips will be dropped from a higher position on to the white sheet]

Interpret: What do you notice about the chips? What can you say about them when you compare them to the results from the previous experiment (when you dropped them at a closer distance to the sheet)? What do you think about these results and your predicted results before the experiment? Can you describe and explain the similarities/differences you see between the actual results and your predictions? Look at each other's results. What do you see in common or different? Can you explain why it happened that way?

Questions:

1. What if you did both experiments with 30 instead of 20 chips. What would the results look like in that case?
2. What is common across all these distributions of chips? Is there any experiment you can think of that the resulting distribution does not have the middle clump?

**Task 3: Dart Game**

Questions: (Show students a picture of a dart board) Have you ever played a dart game? Is this familiar to you? What do you know about the rules and scoring? If you were to play the game, what would you do to get the highest score?

**Task 4: Design Your Own Game**

You will create your own game in which you can give different points for landing near or far from a target. In doing so, you will choose the height at which you drop the chips and the number of chips.

Questions: How did you choose your height? How is it going to affect your game? Why did you pick X amount of chips? How did you determine different regions and assigned points? Is this something you have done before? Play the game.

**Task 5: Gumballs Activity**

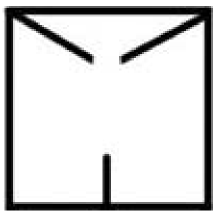
Materials: One gumball machine with a mixture of 50 gumballs (15 yellow, 5 blue, 10 red, 7 green, 5 purple, and 8 pink)

In this gumball machine there is a mixture of gumballs with different colors. Looking at this (you hold it to examine what is inside), can you guess what color gumball you will get? How did you make your guess? How about next (with replacement)?

## Tasks Used in Teaching Episodes 4 and 5

**Task 6: The Split-box**

Materials: The split-box and wooden marbles (in packs of 10, 50, and 100)



10 Individual Marbles: This is a split-box for marbles. You will drop one marble through this funnel at the top and it will go to one of these two compartments (Left and Right) at the bottom. Write your predictions and actual results on the given record sheet for each marble. Before you drop 10 marbles, make a prediction for each one first.

1. Where will the marble go?
2. Why?

After the experiment:

1. What happened?

2. How/why did that happen?

3. What do you think about your predictions and the actual results?

10 Marbles:

1. What is your prediction if we drop all 10 marbles?

2. Will we probably get more on one side or the other? By how much? Or will they be the same?

3. Will they be more likely to go left or right, or are the chances the same for both sides?

4. Is that possible? Why? Why [more on right]?

After the experiment:

1. Why do you have [the same]/[more on left]/[more on right] on each side?

2. What had you said it would be?

3. Why is it the same on both sides? Do you know why?

4. You see it is the same on both sides. Are you surprised?

50 Marbles and 100 Marbles: (The same questions as above)

Prediction questions for experiments with large number of marbles:

1. How will they fall if we drop 50 marbles? 200? 1000?

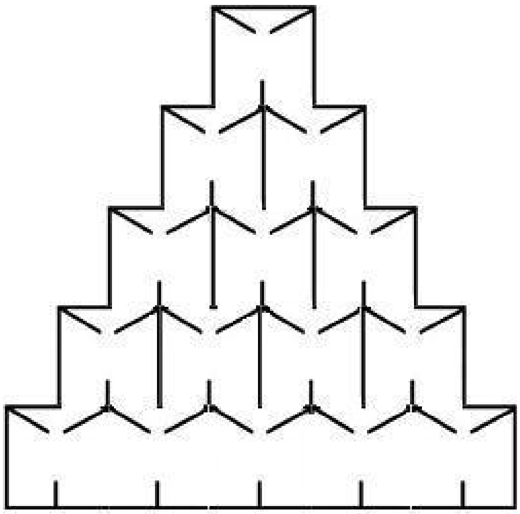
2. If we did it all again, with 500 marbles, would the difference get larger or smaller?

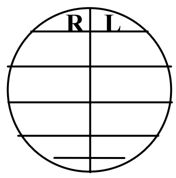
3. Is it more or less regular where there are many? Why? Why not?

**Task 7: Multi-level Split-box Game**

This is a type of board game that you can play with the split-box by dropping a marble. You have some counters representing marbles which start falling from the upper box.  As in the split-box, when the counter meets the divider at each step (on the board), it has to decide whether to go right or left.  Therefore, you will drop the marble in the split-box for each step on the board game and then move your counter according to that result (left or right).  And you need to mark L or R on the counter for each step.  Keep doing this till the counter (or marble) ends up in one of the compartments at the bottom (five times).

Multi-Level Split-Box and the example of a counter:






Prediction: Which compartment do you think the counter will end up? Why?

Experiment: Each of you will take a turn and play at least 3 times. Mark R-L on the counter after each step.

Interpret:

1. What do you notice about the distribution?
2. Why didn't any of them go to this compartment?
3. Why did only few marbles go to that compartment?
4. Where did marbles arrive mostly? Do you wonder why?

5. What do you notice about the marbles in the same compartment? How did they end up in the same place? Did they follow the same path all the way?

Further questions:

1. If we play this with 600 marbles, where will they go? The same in all? Where will there be the most? And the fewest?
2. If we were to find a winner for this game, which compartment should she/he bet before playing the game?

**Tasks Used in Teaching Episodes 6, 7, 8, and 9**

**Task 8: Bears Task**

Materials: 5 red bears and 5 blue bears made from cardboard, paper, and markers.

<u>Goal</u>: Figure out as many different ways as you can to make a bulletin board decoration with five bears. You have five red bears and five blue bears to choose from.

Questions:

1. How many different ways can you choose five bears from these? Is there any other way? Can you do them all? Is there still another way? Are you sure? [Combinations, 6 ways]
2. How many different ways can 4 red bears and 1 blue bear be arranged to make the decoration? Did I ask you the same thing? Are these two questions the same? [Permutations, 5 ways]

**Task 9: Coin Flipping Activity**

Questions:

1. When do we flip a coin? For what purposes?
2. What are the possible outcomes when you flip a coin?

5 coin-tosses: Before starting flipping the coin, predict if it is going to be Heads or Tails. What do you expect in the second trial? How did you make your prediction?

10 coin-tosses: How many Heads-Tails would you expect if you flip a coin 10 times? Record your prediction and give a reason for why you think so. Flip the coin 10 times and record the outcomes. Did the experiment turn out as you predicted? How are the results same? How are the results different?

Prediction question: If you flipped the coin 50 times, how many Heads-Tails would you expect? Explain your reasoning. What if you did it 100/200/1000 times? Explain why.

**Task 10: Spinner Task**

Material: A spinner with 3 equal-sized parts (yellow, blue, and purple)

You see this spinner? It has 3 equal parts with yellow, blue, and purple. You will make predictions for the outcomes if you were to spin this spinner 5, 10, 20, 30, 100, 300, 1500, and 2000 times. Please write your predictions on the record sheet for each number of spins. How did they make your predictions?

**Task 11: Hopping Rabbits Activity**

Suppose there are a number of rabbits on a land and each rabbit can choose to hop only right or left. For each hop, rabbits are just as likely to hop right as left. We want to know where a rabbit is likely to be after 5 hops.

Questions:

1. Do you have any questions so far? Where do you think a rabbit is more likely to be after 5 hops (remember they can only hop right or left)? What if the rabbit begins hopping 5 times again starting from the same place as previously?

2. How about you flip a coin to decide whether a rabbit will hop to right or left and determine where the rabbit will be after five hops by flipping the coin five times. First let's decide which way is Tails or Heads. Can you make a guess where you think a rabbit most/least likely to be after five hops? (We can discuss the possible and impossible outcomes and the reasons)

<u>Simulations with flipping a coin</u>: In your group, flip a coin 5 times to find where the rabbit will be after five hops (each of you will take a turn). Generate results for 10 trials. Mark where it ends up on the number line on your sheet.

Prediction: Where do you think they will be most likely to be?

Questions after the simulation:

1. Did everything turn out to be the way you expected?
2. What do you notice about the results of rabbit's location after five hops?
3. Where is a rabbit more/less likely to be after five hops? Are there equal chances to land on each?
4. Which places are hard or easy to get?
5. Why can they be on only odd numbers?
6. What would you expect if you did this experiment 100 times?

<u>Computer Simulation</u>:

Materials: A Laptop with the computer simulation of *Hopping Rabbits* in NetLogo modified from Wilensky (1998) and projector.

Simulation 1: Conduct simulations with 10/100/500/1000/10000 rabbits.

1. Where do you expect more or less rabbits [before each simulation]?

2. If you do the same thing 100 times, what would you predict [100 rabbits]?

3. Can you tell where the rabbits are most/least likely to be when you look at the simulation distribution?

4. Why did more/less of the rabbits end up there?

Simulation 2: Now watch the individual rabbit 10 times for five hops [using the NetLogo feature of watching a rabbit as it is hopping], but first make your predictions.

1. How did you make you prediction?
2. What happened?
3. Where is it most likely to end up? Why?
4. How many ways are there to end up there?
5. Why do we have more rabbits on 1s than on 3s and few on 5s? You need to convince me to believe you. (Hint: How many different ways to get there? Like you did with the Blue and Red Bears, could you find all the possible ways there are for rabbits after five hops?)

<u>Combinations-Permutations</u>:

1. What are the possible outcomes after five hops?
2. Can you work together to find different ways to get each outcome, such as 1s, 3s, and 5s?
3. How many ways in total did you find to get there?
4. How does it affect the chances of landing on 1s, 3s, and 5s?
5. If you want to calculate the chances to land on 1s and 3s, is the chance of landing on 1s greater than/less than/equal to the chance of landing on 3s?

10 Rabbit-Hops (Computer Simulation):

Prediction: If 10000 rabbits hop 10 times instead of 5, where do you think they are most/least likely to be? Why? How is it going to be similar to or different than 5 hops?

Experiment: Conduct 3 trials with 10000 rabbits with 10 hops.

Interpret: What do you notice about the distribution of rabbits after 10 hops? Where is a rabbit more/less likely to be? Why? Which places are hard or easy to get? Why?

Changing "chance-of-hopping-right = 50%":

Predict-Experiment-Interpret:

1. If you change it from 50% to 75% (and to 25%), what do you expect for the distribution of rabbits?
2. Where will be the most? And the fewest? Why?
3. Did it turn out to be the way you expected?

### Tasks Used in Teaching Episodes 10 and 11

**Task 12: Rolling a Die and Sum of Two Dice**

Predict or guess an outcome when you roll a die.

1. Why did you pick that number?
2. Is one more likely than the other? Why?
3. What are the possible outcomes when you roll a die?
4. Do they all have the same chance of getting rolled?
5. Do you think that there is more chance to roll a certain number than the other ones or there are equal chances to roll any of these numbers?

Consider rolling two dice.  At each trial, you sum the two numbers you roll.

1. What are the possible outcomes? What are the numbers you could get?
2. Which of the following is the most likely result? (1) A total of 6 or 8 (2) A total of 11 or 12?
3. What are all different ways to make each outcome happen?
4. How many different ways can you get a total of 1? 2? 3?....12? More than 12?
5. If you roll two dice 100 times, how many of each sum will you get? Which sum will you get the most/the least?

**Task 13: Galton Box**

Materials: A Laptop with the computer simulation of the Galton Box in NetLogo modified from Wilensky (2002) and the projector.

<u>Simulation 1 (100 balls and 1 row)</u>:

Prediction:

1. If you let 100 balls run down, where will they go?
2. Why more on right or left or the same on both sides?
3. If you do this 10 times again and again, do you always expect more on right or left?

Experiment: Watch the movements of the balls by slowing down the simulation.

Interpret:

1. What do you notice?
2. Why the same on both sides? Why more on the right side or left side?
3. Does it remind you anything you have done before in this class?

Simulation 2 (10 and 100 balls and 5 row):

Prediction:

1. If you increase the number of rows what would you expect of the resulting distribution of the balls? In what ways will it be similar or different?
2. Where will the marbles go?
3. Where will be the most? And the fewest?
4. And if we start again, will it be like this?
5. And if there were no pegs (triangle) in the middle?
6. If you drop more balls, like 100, what would you guess about the distribution of balls at the bottom?

Experiment: Conduct the simulation several times for 10 balls and then 100 balls.

Interpret:

1. Did it turn out to be the way you expected?
2. Does it remind you anything you have done it before in this class? Explain?
3. Why are there more balls in the middle and less on the sides?
4. Is it more or less regular when there are many balls?

Simulation 3 (100 balls and 10 row): (The same questions asked in Simulation 2 above) In this simulation, you will also use other features available in the NetLogo model, such as "shading the path" and "hiding the pile-up." When you turned on shading the path, what do you see? Discuss the change in the color of shading as balls run down. When hiding the pile of balls at the bottom, what can you tell about the distribution of the balls at the bottom?

Changing "chance-of-bouncing-right = 50%":

Predict-Experiment-Interpret:

1. If you change it from 50% to 75% (and to 25%), what do you expect for the distribution of balls?
2. Where will be the most? And the fewest? Why?
3. Did it turn out to be the way you expected?

## APPENDIX E

## DEFINITIONS OF PROBABILISTIC CONCEPTS

**Combinations:** Unordered arrangements of objects are called combinations. The number of combinations of a set of n objects taken r at a time is given by nCr = (n!)/(r!(n - r)!).

**Compound Event:** A compound event consists of two or more simple events.

**Empirical probability:** Empirical probability is based on observation. The empirical probability of an event is the relative frequency of a frequency distribution based upon observation, i.e. P(A) = f / n (f: frequency of observations and n: number of observations)

**Permutations:** A permutation of a set of objects is an arrangement of the objects in a certain order. The total number of permutations of a set of n objects is given by n!.

**Probability of Compound Events:** If, for example, A and B represent two independent events, the probability that both A and B will occur is given by the product of their separate probabilities. The probability that either of the two events A and B will occur is given by the sum of their separate probabilities minus the probability that they will both occur.

**Sample Space:** A sample space is the set of all possible outcomes, i.e. consider the sample space for flipping a coin, {Heads, Tails}.

**The Law of Large Numbers:** In repeated, independent trials with the same probability *p* of success in each trial, the chance of getting any difference (greater than 0) between the percentage of successes (empirical probability) and the probability *p* (theoretical probability) approaches to zero as the number of trials *n* goes to infinity.

**Theoretical Probability:** (Classical probability) When all outcomes in event A are equally likely, the probability of an event occurring is the number of outcomes in the event divided by the number of outcomes in the sample space, i.e. P(A) = (Number of outcomes corresponding to event A) / (Total number of outcomes).